2.9 Reduction of Order

$$y''(t) = f(t, y(t), y'(t)) \qquad (1)$$

Second order equation

Case 1. If $\boxed{y''(t) = f(t, y'(t))}$ (2)

Change variables:

$$v(t) = y'(t) \text{ implies } v'(t) = y''(t) \qquad (3)$$

Thus, (2) is transformed into

$$v'(t) = f(t, v(t)) \qquad \text{First order equ.}$$

Once $v(t)$ is obtained

$$y(t) = \int v(t)\,dt$$

#38) $y'' + t(y')^2 = 0, \quad v = y'$

$$v' + tv^2 = 0 \Rightarrow \frac{v'}{v^2} = -t \text{ or } \frac{dv}{v^2} = -t\,dt$$

$$\int dv : -v^{-1} = -\frac{t^2}{2} + c \Rightarrow \frac{1}{v(t)} = \frac{t^2}{2} + c = \frac{t^2 + 2c}{2} = \frac{t^2 + d}{2}$$

$$\text{or } v(t) = \frac{2}{t^2 + d} \Rightarrow y(t) = \int v(t)\,dt = 2\int \frac{dt}{t^2 + (\sqrt{d})^2}$$

\# 40) $y'' + y' = e^{-t}$

$$v = y' \Rightarrow v' = y''$$

$$v' + v = e^{-t} \Rightarrow \mu(t) = e^{t}$$

$$\frac{d}{dt}[ve^{t}] = e^{-t}e^{t} = 1$$

$$\int dt \quad v(t)e^{t} = t + c \Rightarrow v(t)e^{t} = t + c$$

$$\Rightarrow \boxed{v(t) = te^{-t} + ce^{-t}}$$

$$\Rightarrow \quad y(t) = \int v(t)\,dt$$

$$= \int (te^{-t} + ce^{-t})\,dt = \int \underset{f}{t}\,\underset{g'}{e^{-t}}\,dt + \int ce^{-t} + d$$

Then, $y(t) = -te^{-t} + \int e^{-t}\,dt - ce^{-t} + d$

or $\quad y(t) = -te^{-t} - e^{-t} - ce^{-t} + d$

or $\quad y(t) = -te^{-t} - (1+c)e^{-t} + d = -te^{-t} + \tilde{c}e^{-t} + d.$

So $\quad \boxed{y(t) = -te^{-t} + \tilde{c}e^{-t} + d}$

Case 2 :

$$y'' = f(y, y') \quad \text{"}t\text{" does not appear.}$$

If $v(t) = y'(t) \Rightarrow \dfrac{dv}{dt} = y''(t) = f(y, v)$

Assuming $v(t) = v(y(t))$

$$y''(t) = \frac{dv}{dt} = \frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{=v} = f(y, v)$$

or $\boxed{v\dfrac{dv}{dy} = f(y, v)}$

Once $v(y)$ has been obtained

$$\frac{dy}{dt}(t) = v(y(t))$$

or $\boxed{\dfrac{dy}{v(y)} = dt}$

#46 $y y'' - (y')^3 = 0$

$\Rightarrow y'' = \frac{1}{y}(y')^3$

If $v = y'$ and $v\frac{dv}{dy} = y''$

$\Rightarrow \quad v\frac{dv}{dy} = \frac{v^3}{y} \Rightarrow \frac{1}{v^2}\,dv = \frac{1}{y}\,dy$

$\int : \quad -v^{-1} = \ln|y| + C_1$

$\frac{1}{v} = -\ln|y| + C_2 \Rightarrow v(y) = \frac{1}{C_2 - \ln|y|}$, $C_2 = -C_1$

Now, $\frac{dy}{dt}(t) = v(y(t))$

or $\frac{dy}{v} = dt \Rightarrow (C_2 - \ln|y|)\,dy = dt$

$\Rightarrow \int : \quad C_2 y - (y\ln|y| - y) = t + C_3$

$\Rightarrow \boxed{-y\ln|y| + y - C_2 y - t = C_3}$

or $\boxed{y\ln|y| - y + C_2 y + t = C_3}$ ✓