3.4 Repeated Roots. Reduction of Order.

Consider

$$y''(t) - 6y'(t) + 9y(t) = 0 \qquad (1.1)$$

Charact. polyn: $r^2 - 6r + 9 = 0$

$$(r-3)^2 = 0 \Rightarrow r_1 = r_2 = 3.$$

So only Soln. $\boxed{y_1(t) = e^{3t}.}$

a second soln: $\boxed{y_2(t) = t\,e^{3t}}$

Verification: $y_2' = e^{3t} + 3t\,e^{3t}$

$$y_2'' = \underbrace{3e^{3t} + 3e^{3t}}_{6e^{3t}} + 9t\,e^{3t}.$$

Subst. into (1.1)

$$\cancel{6e^{3t}} + \cancel{9t}e^{3t} - \cancel{6e^{3t}} - \cancel{18t}e^{3t} + \cancel{9t}e^{3t} = 0$$

How can we obtain $y_2(t) = t\,e^{3t}$?

Key idea: look for a soln: as: $\boxed{y_2(t) = v(t)\,e^{3t}}$ (1.2)

then, $\boxed{y_2'(t) = v'e^{3t} + 3v\,e^{3t}}$ (1.3)

$$y_2''(t) = v''e^{3t} + 3v'e^{3t} + 3v'e^{3t} + 9v\,e^{3t}$$

or $\boxed{y_2''(t) = (v'' + 6v' + 9v)\,e^{3t}}$ (1.4)

Subst. of (1.2)–(1.4) into (1.1)

$$e^{3t}\left[v'' + \cancel{6v'} + \cancel{9v} - 6(\cancel{v'} + \cancel{3v}) + \cancel{9v}\right] = 0$$

Since $e^{3t} \neq 0 \Rightarrow v''(t) = 0 \Rightarrow v(t) = c_1 + c_2 t$

Thus, a second soln. is given by

$$y_2(t) = v(t)\, e^{3t} = (c_1 + c_2 t)\, e^{3t}$$

We choose the simplest of these

$$\underline{y_2(t) = t\, e^{3t}} \qquad (2.1)$$

This construction of a second soln. from one already known can also be applied for the more general case:

$$y''(t) + p(t)\, y' + q(t)\, y = 0 \qquad (2.2)$$

We will work an example first. Then, we will present the construction for (2.2)

Sect 3.4

\# 28) $(x-1)y'' - xy' + y = 0, \quad x > 1 \qquad (1)$

If one solution is $y_1(x) = e^x$

find a 2^{nd} Soln. $\to$ Answ: $y_2(x) = x$ How can we find this Soln?

Verify it!

We look for a 2^{nd} Soln. in the form:

$$y_2(x) = v(x)e^x.$$

Then,

$$y_2' = v'e^x + ve^x$$

$$y_2'' = v''e^x + \underbrace{v'e^x + v'e^x} + ve^x$$

Subst. into (1)

$$e^x\left[(x-1)(v'' + 2v' + v) - x(v' + v) + v\right] = 0$$

or

$$(x-1)v'' + \left[2(x-1) - x\right]v' + \left[\cancel{(x-1) - x + 1}^{\,0}\right]v = 0$$

$$(x-1)v'' + (2(x-1) - x)v' = 0$$

If $\boxed{u = v'}$

$$(x-1)u' + (x-2)u = 0$$

$$\frac{du}{dx} = -\frac{(x-2)}{x-1}u \Rightarrow \frac{du}{u} = -\frac{(x-2)}{x-1}dx$$

$$\int : \quad \ln|u| = -\int\left[\left(\frac{x}{x-1}\right) + \left(\frac{-2}{x-1}\right)\right]dx = -\int\left(1 - \frac{1}{x-1}\right)dx$$

Now, $\displaystyle\int \frac{x}{x-1}dx = \int\left(1 + \frac{1}{x-1}\right)dx = x + \ln|x-1|$

$\therefore$

$$\ln|u| = -(x + \ln|x-1|) + 2\ln|x-1| + C = -x + \ln|x-1| + C$$

$$u(x) = C_2 e^{-x} \cdot e^{\ln|x-1|} = C_2 e^{-x}|x-1| = C_3 e^{-x}(x-1).$$

$\Rightarrow$

$$v'(x) = e^{-x}(x-1) \Rightarrow v(x) = \int e^{-x}(x-1)dx = -xe^{-x} + C$$

$\Rightarrow$

and a second soln:

$$y_2(x) = v(x)\, e^{x} = \left(-xe^{-x} + c\right)e^{x} = -x + ce^{x}$$

Choosing $c = 0$ and multiplying by (-1) $y_2(x)$ $\boxed{y_2(x) = x}$

$$\int \underset{g'}{e^{-x}}\, \underset{f}{x}\, dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}.$$

In general, for (2.2)

we look for a second solution in the form:

$$y_2(t) = v(t)\, y_1(t)$$

Then,

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + \underbrace{v' y_1' + v' y_1'}_{2v'y_1'} + v y_1''$$

Substitution into (2.2)

$$v''y_1 + 2\underline{v'}y_1' + vy_1'' + p(t)\left(\underline{v'}y_1 + vy_1'\right) + q(t)vy_1 = 0$$

or

$$v''y_1 + \left(2y_1' + p(t)y_1\right)v' + \left(y_1'' + p(t)y_1' + q(t)y_1\right)^{\nearrow 0} v = 0$$

Thus,

$$\boxed{y_1(t)\,v''(t) + \left(2y_1'(t) + p(t)y_1\right)v'(t) = 0} \quad (3.1)$$

Introduce: $u(t) = v'(t)$

Then (3.1) reduces to

$$\boxed{y_1(t)\,u'(t) + \left(2y_1'(t) + p(t)y_1\right)u(t) = 0} \quad (3.2)$$

Which is a 1st order ODE. (linear and Separable)

Once we solve for (3.2), we will have a Second Soln. for (2.2.1) $y_2(t)$. It can be proved that $y_1(t)$ and $y_2(t)$ form a fund. set.

In particular, if we are solving a constant coefficients equation:

$$\boxed{ay'' + by' + cy = 0} \qquad (4.1)$$

or $$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \qquad p(t) \equiv \frac{b}{a}, \quad q(t) \equiv \frac{c}{a}$$

and $y_1(t) = e^{r_1 t}$ $\quad r_1 = -\frac{b}{2a}$ is the only root. (real)
only solution of exp. form. $\underline{\underline{e}}^{rt}$.

then by applying reduction of order, we arrive to equ. (3.1) where

$$2y_1'(t) + p(t)y_1(t) \equiv \cancel{2}\left(\frac{-b}{2\cancel{a}}\right)e^{-\frac{b}{2a}t} + \frac{b}{a}e^{-\frac{b}{2a}t} = 0.$$

Therefore, (3.1) reduces to

$$v''(t)\,y_1(t) = v''(t)\,e^{r_1 t} = 0 \Rightarrow$$

$$v''(t) = 0 \Rightarrow \quad v(t) = C_1 + C_2 t$$

$$\therefore \quad y_2(t) = (C_1 + C_2 t)e^{r_1 t} = \underbrace{C_1 e^{r_1 t}}_{\text{eliminate this or combine with first soln.}} + C_2 t e^{r_1 t}$$

or $\boxed{y_2(t) = t e^{r_1 t}}$

The Solns. $y_1(t) = e^{r_1 t}$ and $y_2(t) = t e^{r_1 t}$ form a fundamental set.

In fact,

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & t e^{r_1 t} \\ r_1 e^{r_1 t} & e^{r_1 t} + r_1 t e^{r_1 t} \end{vmatrix}$$

$$= e^{2r_1 t} + \cancel{r_1 t e^{2r_1 t}} - \cancel{r_1 t e^{2r_1 t}} = e^{2r_1 t} \neq 0.$$

Therefore, the general Soln. of (4.1) is given by

$$\boxed{y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}} \qquad (5.1).$$