3.5 Method of Undetermined Coefficients.

Nonhomogeneous Equation.

Consider

$$\boxed{y''(t)+p(t)y'(t)+q(t)y(t)=g(t)} \qquad (1)$$

and its corresponding homogeneous equation.

$$\boxed{y''+p(t)y'+q(t)y=0} \qquad (2)$$

with $p(t)$, $q(t)$ and $g(t)$ continuous on (α,β)

Theorem The general Soln. of (1) can be written as

$$\boxed{y(t)=C_1y_1(t)+C_2y_2(t)+y_p(t)} \qquad (3)$$

where $\{y_1,y_2\}$ form a fundamental set for (2) and y_p is a particular solution of (1).

Proof. The function (3) is a solution of (1) since

$$y''+p(t)y'+q(t)y=$$

$$C_1y_1''+C_2y_2''+y_p''+p(t)\left(C_1y_1'+C_2y_2'+y_p'\right)$$

$$+q(t)\left(C_1y_1+C_2y_2+y_p\right)=$$

$$C_1(\cancelto{0}{y_1''+py_1'+qy_1})+C_2(\cancelto{0}{y_2''+py_2'+y_2})$$

$$+y_p''+py_p'+qy_p=0+0+g(t)=g(t).$$

On the other hand, if $\phi(t)$ is an arbitrary Soln of (1), then

$$\phi(t) = d_1 y_1 + d_2 y_2 + y_p, \text{ for particulars } d_1 \text{ and } d_2.$$

In fact, $\phi(t) - y_p(t)$ is a Soln. of (2)

$$(\phi(t) - y_p(t))'' + p(t)(\phi(t) + y_p(t))' + q(t)(\phi(t) + y_p(t))$$
$$= (\phi'' + p\phi' + q\phi) - (y_p'' + p y_p' + q y_p) =$$
$$g(t) \quad - \quad g(t) = 0$$

Therefore, $\phi(t) - y_p(t) = d_1 y_1(t) + d_2 y_2(t)$ (4)

Since y_1 and y_2 form a fund. set. for (2)

Solving for $\phi(t)$ in (4)

$$\boxed{\phi(t) = d_1 y_1 + d_2 y_2 + y_p.} \qquad (5)$$

As a result, to find the general Soln. of (1) we need

1) Find a fundamental set for (1).
2) Find a particular Soln. of (2).

Method of Undetermined Coefficients for Linear, Second order, constant coefficients and nonhomogeneous ODE.

$$\boxed{y'' - 2y' - 3y = 6e^{4t}} \qquad (M.1).$$

Charact. equation: $r^2 - 2r - 3 = 0$

roots: $(r-3)(r+1) = 0$

$r_1 = -1 \Rightarrow y_1 = e^{-t}$, $r_2 = 3 \Rightarrow y_2 = e^{3t}$

Key idea: look for Soln. as: $\boxed{y_p(t) = A e^{4t}}$

Subst. into (M.1)

$$y_p'' - 2y_p' - 3y_p = e^{4t}(16A - 8A - 3A)$$

$$= 5A\, e^{4t}$$

For $y_p(t) = Ae^{4t}$ to be a solution of (M.1)

$5A\, e^{4t} = 6e^{4t}$ should be verified.

then $A = 6/5$

and $\boxed{y_p(t) = 6/5\, e^{4t}}$ is a particular Soln. of (M.1)

$$\boxed{y'' - 2y' - 3y = 3\cos(6t)} \qquad (M.2)$$

First trial, $y_p = A\cos(6t)$

$y_p' = -6A\sin(6t)$, $y_p'' = -36A\cos(6t)$

Subst. into (M.2)

$$-36A\cos(6t) + 12A\sin(6t) - 3A\cos(6t)$$

$$-39A\cos(6t) + 12A\sin(6t) \overset{\text{If soln. of (M.2)}}{=\!=} 3\cos(6t).$$

for all t.

If $t=0$

$$-39A = 3 \Rightarrow A = -3/39.$$

If $t = \pi/12$

$$12A = 0 \Rightarrow A = 0 \quad \text{Contradiction!}$$

Therefore, there are no solutions as $y_p = A\cos(6t)$.

<u>Key idea</u>: Look for soln. as

$$\boxed{y_p = A\cos(6t) + B\sin(6t)}$$

Subst. into (M.2) leads to

$$(-36A - 12B - 3A)\cos(6t) + (-36B + 12A - 3B)\sin(6t)$$

$$= 3\cos(6t)$$

The functions $\cos(6t)$ and $\sin(6t)$ are linearly independent, then

$$\begin{array}{ll} \cos(6t): & \left\{ \begin{array}{l} -39A-12B=3 \\ -39B+12A=0 \end{array} \right. \\ \sin(6t): & \end{array} \Rightarrow A=\frac{-13}{185}, \quad B=\frac{-4}{185}$$

and

$$\boxed{y_p = \frac{-13}{185}\cos(6t) - \frac{4}{185}\sin(6t)}$$

The general soln. of (M.2) is given by

$$\boxed{y(t) = C_1 e^{-t} + C_2 e^{3t} - \frac{13}{185}\cos(6t) - \frac{4}{185}\sin(6t)}$$

$$\boxed{y'' - 2y' - 3y = 2t^2 + 3} \quad (M.3)$$

Key idea: $\boxed{y_p(t) = At^2 + Bt + C}$ $\quad y_p' = 2At + B$

$y_p'' = 2A$

Subst. into (M.3)

$$2A - 4At - 2B - 3At^2 - 3Bt - 3C = 2t^2 + 3$$

or $-3At^2 + (-4A-3B)t + (2A-2B-3C) = 2t^2 + 3$

Then, t^2: $-3A = 2 \Rightarrow A = -2/3$

t: $-4A - 3B = 0 \Rightarrow B = 8/9$

1: $2A - 2B - 3C = 0 \Rightarrow C = \frac{-55}{27}$

So general soln:

$$\boxed{y(t) = c_1 e^{-t} + c_2 e^{3t} - \frac{2}{3}t^2 + \frac{8}{9}t - \frac{55}{27}}$$

RHS Product of exponentials, sine/cosine, and/or polynomials

$$\boxed{y'' - 2y' - 3y = 5e^{2t}\sin t} \qquad (M.4)$$

Key idea: Look for solns as

$$\boxed{y_p(t) = Ae^{2t}\sin t + Be^{2t}\cos t}.$$

Longer computation. At the end arrive to

$$\begin{array}{ll} e^{2t}\sin t: & \\ e^{2t}\cos t: & \end{array} \begin{cases} -2B - 4A = 5 \\ -4B + 2A = 0 \end{cases} \Rightarrow A = -1, \ B = -1/2.$$

General Solution $\boxed{y(t) = c_1 e^{-t} + c_2 e^{3t} - e^{2t}\sin t - \frac{1}{2}e^{2t}\cos t}$

Ask for $\boxed{y'' - 2y' - 3y = (t^3 + t)\begin{cases} e^{2t} \\ \sin t \\ e^t \sin t \end{cases}}$

Consider

$$\boxed{y''-2y'-3y=4e^{3t}} \qquad (M.5)$$

$$\boxed{y_p = Ae^{3t}} \Rightarrow y_p' = 3Ae^{3t}, \quad y_p'' = 9Ae^{3t}$$

Subst.

$$(\cancel{9A}-\cancel{6A}-\cancel{3A})e^{3t} = 4e^{3t}$$

$$0=4!$$

Problem is $y_p = Ae^{3t}$ is a Soln. of homog. equ.

Consider a Similar 1st order problem:

$$y'+y = 2e^{-t} \qquad (7.1)$$

a Soln of the corresponding homog. equation: $y_h(t) = e^{-t}$.

I.F. $\mu(t) = e^{t}$

We can solve (7.1). In fact,

$$\frac{d}{dt}[ye^{t}] = 2e^{-t}\cdot e^{t} = 2$$

$$\Rightarrow y(t)e^{t} = 2t + C$$

$$\Rightarrow \boxed{y(t) = 2te^{-t} + Ce^{-t}} \quad \text{General soln.}$$

This Suggests: $y_p(t) \overset{\text{for (M.5)}}{=\!=} Ate^{3t}$

$$y_p' = 3Ate^{3t} + Ae^{3t}$$

$$y_p'' = 9Ate^{3t} + 6Ae^{3t}$$

Subst. into (M.5)

$$\overbrace{(9A - 6A - 3A)}^{0}\, t e^{3t} + \overbrace{(6A - 2A)}^{=4A} e^{3t} = 4e^{3t}$$

Typical.

$$A = 4 \Rightarrow \boxed{A = 1}$$

General Soln.: $\boxed{y(t) = C_1 e^{-t} + C_2 e^{3t} + t e^{3t}}$

Consider $\boxed{y'' - 6y' + 9y = 4e^{3t}}$ (M.6)

$$r^2 - 6r + 9 = 0 \Leftrightarrow (r-3)^2 = 0$$

$$y_1(t) = e^{3t}, \quad y_2(t) = te^{3t}$$

Then, $\boxed{y_p(t) = At^2 e^{3t}}$

$$y_p' = 2Ate^{3t} + 3At^2e^{3t}, \quad y_p'' = 2Ae^{3t} + 6Ate^{3t} + 6Ate^{3t} + 9At^2e^{3t}$$

$$6Ate^{3t} + 6Ate^{3t} = 12Ate^{3t}$$

Subst. into (M.6)

$$2Ae^{3t} + \cancel{12Ate^{3t}} + \cancel{9At^2e^{3t}} - \cancel{12Ate^{3t}} - \cancel{18At^2e^{3t}} + \cancel{9At^2e^{3t}} = 4e^{3t}$$

$$\therefore \quad 2A = 4 \Rightarrow \boxed{A = 2}$$

or $\boxed{y_p = 2t^2e^{3t}}$

and General Soln: $\boxed{y(t) = C_1 e^{3t} + C_2 te^{3t} + 2t^2 e^{3t}}$

Consider the following:

a) $y'' - 2y' - 3y = 4t^5 e^{3t}$, $\quad y_1(t) = e^{-t}$; $y_2(t) = e^{3t}$

$y_p(t) = t\left(At^5 + Bt^4 + ct^3 + dt^2 + et + f\right)e^{3t}$

b) $y'' - 6y' + 9y = 4t^3 e^{3t}$

$y_p(t) = t^2\left(At^3 + Bt^2 + ct + d\right)e^{3t}$.

c) $y'' + y = \cos t$

Charact. Equ: $r^2 + 1 = 0$, $r_{1,2} = \pm i$ $\quad y_1(t) = \cos t$, $\quad y_2(t) = \sin t$

$y_p(t) = tA\cos t + tB\sin t$.

Go to maple: $y(t) = C_1\cos t + C_2\sin t + -\cos t + \frac{t}{2}\sin t$

d) $y'' + y = t^3\cos t$

$\boxed{y_p(t) = t\left(At^3 + Bt^2 + Ct + D\right)\cos t + t\left(Et^3 + Ft^2 + Gt + H\right) * \sin t}$

$\boxed{y(t) = C_1\cos t + C_2\sin t - 3t\cos t - \frac{3}{8}t^2\sin t + \frac{1}{4}t^3\cos t + \frac{t^4}{8}\sin t.}$

Nonhomog term is a sum of previous nonhomog terms (8)

(6) $y'' - 2y' - 3y = 6e^{4t} + 3\cos(6t) + 2t^2 + 3 + 5e^{2t}\sin t.$

$y_p(t) = \frac{6}{5}e^{4t} - \frac{13}{185}\cos 6t - \frac{4}{185}\sin 6t - \frac{2}{3}t^2 + \frac{8}{9}t - \frac{55}{27} - e^{2t}\sin t - \frac{1}{2}e^{2t}\cos t$

Based on this result: Show MAPLE!

Let $y_{p_1}(t)$ be a soln. of
$$ay'' + by' + cy = g_1(t)$$

and $y_{p_2}(t)$ be a soln of
$$ay'' + by' + cy = g_2(t)$$

$\Rightarrow$ $\boxed{y_p(t) = y_{p_1} + y_{p_2}}$ is a soln of
$$ay'' + by' + cy = g_1(t) + g_2(t)$$

Proof:-
$$(y_{p_1} + y_{p_2})' = y_{p_1}' + y_{p_2}'$$
$$(y_{p_1} + y_{p_2})'' = y_{p_1}'' + y_{p_2}''$$

$\Rightarrow$
$$a(y_p'') + b\,y_p' + c\,y_p =$$
$$= a\,y_{p_1}'' + a\,y_{p_2}'' + b\,y_{p_1}' + b\,y_{p_2}' + c\,y_{p_1} + c\,y_{p_2} =$$
$$= (a\,y_{p_1}'' + b\,y_{p_1}' + c\,y_{p_1}) + (a\,y_{p_2}'' + b\,y_{p_2}' + c\,y_{p_2})$$
$$= g_1(t) + g_2(t) = g(t) \checkmark$$

7 (Example 5 book) (Similar to our 5)

$$y'' + 4y = 3\cos 2t$$

$$r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm 2i$$

$$\Rightarrow \quad y_1(t) = \cos 2t, \qquad y_2(t) = \sin 2t.$$

$$\boxed{y_p(t) = At\cos 2t + Bt\sin 2t}$$

Subst. in equ. leads to $\boxed{y_p(t) = \frac{3}{4} t \sin(2t)}$

Summary

$$ay'' + by' + cy = g(t) \qquad (1)$$

$$g(t) = g_1(t) + \dots + g_n(t) \qquad (2)$$

then (1) can be divided in n nonhomog. ODE.

$$ay'' + by' + cy = g_i(t) \qquad i = 1, \dots, n \qquad (3)$$

Each ODE in (3) and consequently (1) can be solved if $g_i(t)$ is of the type discussed above and in this case $y_p(t)$ has a particular form depending on $g_i(t)$.

TABLE 3.6.1 (BOOK)

$g_i(t)$	$y_{p_i}(t)$
$P_n(t) = a_0 t^n + \dots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)$
$P_n(t)\, e^{\alpha t}$	$t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)\, e^{\alpha t}$
$P_n(t)\, e^{\alpha t} \begin{cases} \sin\beta t \\ \cos\beta t \end{cases}$	$t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)\, e^{\alpha t}\cos\beta t$ $+\, t^s(B_0 t^n + B_1 t^{n-1} + \dots + B_n)\, e^{\alpha t}\sin\beta t$

$s = 0, 1, 2$ such that no term in $y_{p_i}(t)$ is a solu. of the corresp. homog. equ.

9'

$$\boxed{y'' - 2y' + y = te^t}$$

Interesting case of repetition! $(r-1)^2=0$

$y_1 = e^t$

$y_2 = te^t$

$$\boxed{y_p(t) = t^2(A_0 + A_1 t)e^t = (A_0 t^2 + A_1 t^3)e^t}$$

$$y_p' = (2A_0 t + 3A_1 t^2 + A_0 t^2 + A_1 t^3)e^t$$

$$y_p' = (2A_0 t + (3A_1 + A_0)t^2 + A_1 t^3)e^t$$

$$y_p'' = (2A_0 + (6A_1 + 2A_0)t + 3A_1 t^2 + 2A_0 t + (3A_1 + A_0)t^2 + A_1 t^3)e^t$$

$$= (2A_0 + (6A_1 + 4A_0)t + (6A_1 + A_0)t^2 + A_1 t^3)e^t$$

$$y_p'' - 2y_p' + y_p = 2A_0 e^t + (6A_1 + 4A_0 - 4A_0)te^t$$

$$+ (\cancel{6A_1} + \cancel{A_0} - \cancel{6A_1} - \cancel{2A_0} + \cancel{A_0})t^2e^t$$

$$+ (A_1 - \cancel{2A_1} + A_1)t^3e^t$$

$$A_1 = \frac{1}{6}$$

$$\boxed{y_p = \frac{1}{6}t^3e^t}$$

$$y_p' = \left(\frac{1}{2}t^2 + \frac{1}{6}t^3\right)e^t$$

$$y_p'' = \left(t + \frac{1}{2}t^2 + \frac{1}{2}t^2 + \frac{1}{6}t^3\right)e^t$$

$$y_p'' - 2y_p' + y_p =$$

$$\Rightarrow te^t + (1 - 1)t^2e^t + \left(\frac{1}{6} - \frac{2}{6} + \frac{1}{6}\right)t^3e^t$$

Notice