3.6 Variation of Parameters.

Consider

$$\boxed{y''(t) + p(t)y'(t) + q(t)y(t) = g(t)} \quad (1.1)$$

Assume $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of (1.1) (they form a fundamental set).

We will look for solutions as:

Key Idea: $$\boxed{y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)} \quad (1.2)$$

where $u_1(t)$ and $u_2(t)$ are unknown functions to be determined.

Procedure:

Step 1: obtain derivatives of y_p (y_p' and y_p'')

Step 2: Assume: $u_1'y_1 + u_2'y_2 = 0$

Step 3: Substitute y_p, y_p' and y_p'' into (1.1)

Step 4: obtain a linear system of equations for u_1' and u_2' and solve it

Step 5: Integrate u_1' and u_2' and subst. into (1.2) To obtain $y_p(t)$.

$$q * \left(y_p = u_1 y_1 + u_2 y_2 \right)$$

key idea.

$$p * \left(y_p' = u_1 y_1' + u_2 y_2' \right), \quad \text{assuming: } \boxed{u_1' y_1 + u_2' y_2 = 0}$$

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

Then,

$$y_p'' + p y_p' + q y_p =$$

$$u_1 \overbrace{\left(y_1'' + p y_1' + q y_1 \right)}^{0} + u_2 \overbrace{\left(y_2'' + p y_2' + q y_2 \right)}^{0}$$

$$+ u_1' y_1' + u_2' y_2' = g(t)$$

As a result,

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 & \text{(V.1)} \\ u_1' y_1' + u_2' y_2' = g(t) & \text{(V.2)} \end{cases}$$

Consider this as a system of two equations with two unknowns: u_1' and u_2'.

Solving the system for u_1' and u_2':

$$u_1' y_1 = -u_2' y_2 \Rightarrow u_1' = -\frac{y_2}{y_1} u_2' \qquad (V.3)$$

Subst. into (V.2)

$$-\frac{y_2}{y_1} u_2' y_1' + u_2' y_2' = g$$

$$\therefore \quad u_2'\left(y_2' - \frac{y_2 y_1'}{y_1}\right) = g$$

or $$u_2'\left(\frac{y_1 y_2' - y_2 y_1'}{y_1}\right) = g$$

$$\Rightarrow \boxed{u_2' = \frac{y_1 g}{y_1 y_2' - y_2 y_1'} = \frac{y_1 g}{W(y_1, y_2)}}$$

and from (V.3) $$\boxed{u_1' = -\frac{y_2}{\cancel{y_1}}\left(\frac{\cancel{y_1} g}{W(y_1, y_2)}\right) = \frac{-y_2 g}{W(y_1, y_2)}}$$

Finally,

$$\boxed{\begin{aligned} u_1(t) &= -\int \frac{y_2(t) g(t)}{W(y_1, y_2)(t)} dt \\ u_2(t) &= \int \frac{y_1(t) g(t)}{W(y_1, y_2)(t)} dt. \end{aligned}}$$

Sect 3.6

#)

$$\boxed{y'' + 4y' + 4y = x^{-2} e^{-2x},} \quad x > 0. \tag{4.1}$$

Charact. equ.: $(r+2)^2 = 0$

Fundamental Set: $\boxed{y_1(x) = e^{-2x},}$ $\boxed{y_2(x) = x e^{-2x}}$

Applying the previous procedure, we propose a particular solution of the form:

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

or

$$\boxed{y_p(x) = u_1(x)\, e^{-2x} + u_2(x)\, x\, e^{-2x}.} \tag{4.2}$$

With $u_1(x)$ and $u_2(x)$ unknown functions.

The key assumption is

$$\boxed{e^{-2x} u_1' + x e^{-2x} u_2' = 0} \tag{4.3}$$

Jumping to <u>step 4</u>, we obtain the following linear system of equations for u_1' and u_2',

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = x^{-2} e^{-2x} \end{cases}$$

$$y_1' = -2e^{-2x}$$
$$y_2' = e^{-2x}(1-2x)$$

or $$\begin{cases} e^{-2x}\left(u_1' + x\, u_2' \right) = 0 \\ \cancel{e^{-2x}}\left(-2u_1' + (1-2x)\, u_2' \right) = x^{-2} \cancel{e^{-2x}} \end{cases}$$

or $$\boxed{\begin{cases} u_1' + x\, u_2' = 0 \\ -2u_1' + (1-2x)\, u_2' = x^{-2} \end{cases}} \qquad (5.1)$$

Solving for u_1' and u_2':

$$\boxed{u_1' = -x\, u_2'} \Rightarrow -2\,(-x\,u_2') + (1-2x)\,u_2' = x^{-2}$$

$$\cancel{2x u_2'} + u_2' - \cancel{2x u_2'} = x^{-2}$$

$$u_2' = x^{-2} \Rightarrow \boxed{u_2(x) = -x^{-1}} + c^{=0}$$

Thus, $u_1' = -x\,u_2' = -x^{-1} \Rightarrow \boxed{u_1(x) = -\ln x} + d^{=0}$

Alternative using Kramer's rule:

$$\begin{cases} u_1' + x\,u_2' = 0 & (6) \\ -2u_1' + (1-2x)\,u_2' = \frac{1}{x^2} & (7) \end{cases}$$

This system can be considered a linear system of equations for the unknowns u_1' and u_2'.

It has a unique soln. if $\Delta \neq 0$

$$\Delta = \begin{vmatrix} 1 & x \\ -2 & 1-2x \end{vmatrix} = 1-2x+2x = 1 \neq 0. \text{ for all } x.$$

and using Cramer's rule

$$u_1' = \frac{\begin{vmatrix} 0 & x \\ 1/x^2 & 1-2x \end{vmatrix}}{\begin{vmatrix} 1 & x \\ -2 & 1-2x \end{vmatrix}} = \frac{-\frac{1}{x}}{1} = -\frac{1}{x} \qquad (8)$$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ -2 & 1/x^2 \end{vmatrix}}{\begin{vmatrix} 1 & x \\ -2 & 1-2x \end{vmatrix}} = \frac{\frac{1}{x^2}}{1} = \frac{1}{x^2} \qquad (9)$$

Integrating (8) and (9).

$$u_1(x) = -\int \frac{1}{x}\,dx \overset{x>0}{=} -\ln(x) + C_1,$$

$$u_2(x) = \int \frac{1}{x^2}\,dx = -x^{-1} + C_2$$

Since, we are looking for particular u_1 and u_2 we choose $C_1 = 0$, $C_2 = 0$., so $\boxed{\begin{aligned} u_1(x) &= -\ln x \\ u_2(x) &= -\frac{1}{x} \end{aligned}}$

Finally, the particular solution we are looking for is given by

$$y_p(x) = u_1(x)\,e^{-2x} + u_2(x)\,x\,e^{-2x} =$$
$$= -\ln(x)\,e^{-2x} - \frac{1}{x}\cdot x\,e^{-2x} = -\ln(x)\,e^{-2x} - e^{-2x}$$

Since e^{-2x} is a Soln. of the homogeneous equation corresponding to (1)

$$\boxed{y_p(x) = -\ln(x)\,e^{-2x}} \quad (10)$$

is a particular Soln.

General Solution: $\boxed{y(x) = C_1 e^{-2x} + C_2 x e^{-2x} - \ln x\, e^{-2x}}$

Thm 3.7.1

Consider
$$\boxed{y'' + p(t)\,y' + q(t)\,y = g(t)} \qquad (8.1)$$
with $p(t)$, $q(t)$ and $g(t)$ continuous on (α, β).

If $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions for the corresponding homogeneous equation,
$$\boxed{y'' + p(t)\,y' + q(t)\,y = 0,} \qquad (8.2)$$
then a particular solution of (8.1) is given by
$$\boxed{y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)} \qquad (8.3)$$

where
$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(y_1, y_2)(t)}\,dt$$

and
$$u_2(t) = \int \frac{y_1(t)\,g(t)}{W(y_1, y_2)(t)}\,dt.$$