

Summarizing example 1 in book

Spring-mass system mathematical model.

$$m u''(t) + \gamma u'(t) + K u(t) = F_{\text{ext}}(t) \quad (S.1)$$

$$\text{Weight} = mg = 4 \text{ lb} \Rightarrow \boxed{m = \frac{4}{32} = \frac{1}{8}}$$

$$\text{Displacement from natural length } l : \boxed{L = 2 \text{ in} = \frac{1}{6} \text{ ft.}}$$

$$\text{Drag force : } F_d = 6 \text{ lb when } u'(t) = 3 \text{ ft/s}$$

$$\text{then } \gamma u' = 6 \text{ or } \boxed{\gamma = 2}$$

Substituting into (S.1)

$$\frac{1}{8} u'' + 2 u' + K u = 0 \quad (S.2)$$

Now, K can be computed from the equilibrium relationship

$$mg = KL \quad (\text{Spring-mass system at rest})$$

$$(u \equiv 0, u' \equiv 0, u'' \equiv 0)$$

Then,

$$K = \frac{mg}{L} = \frac{4}{1/6} = 24$$

Finally, IVP to be solved is given by

$$\text{IVP} \quad \begin{cases} u'' + 16 u' + 192 u = 0 \\ u(0) = 6 \text{ in} = \frac{1}{2} \text{ ft.}, \quad u'(0) = 0 \end{cases} \quad (S.3)$$

Charact. polynomial:

$$r^2 + 16r + 192 = 0$$

$$r_{1,2} = -8 \pm 8\sqrt{2}i$$

Then, $u(t) = A e^{-8t} \cos(8\sqrt{2}t) + B e^{-8t} \sin(8\sqrt{2}t)$

Using I.C.s.

$$u(t) = \frac{1}{2} e^{-8t} \cos(8\sqrt{2}t) + \frac{1}{4} \sqrt{2} e^{-8t} \sin(8\sqrt{2}t). \quad (5.4).$$

Show graph in MAPLE.

Damped Free Vibration. No external force.

$$\boxed{m u'' + \gamma u' + k u = 0} \quad (5.5)$$

, $m, \gamma, k > 0$

Charact. equ.: $m r^2 + \gamma r + k = 0$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

If (I) $\gamma^2 - 4mk > 0 \Rightarrow u = A e^{r_1 t} + B e^{r_2 t}$

(II) $\gamma^2 - 4mk = 0 \Rightarrow u = A e^{r_1 t} + B t e^{r_1 t}$

(III) $\gamma^2 - 4mk < 0 \Rightarrow u = A e^{-\frac{\gamma}{2m} t} \cos\left(\frac{\sqrt{4mk - \gamma^2}}{2m} t\right) + B e^{-\frac{\gamma}{2m} t} \sin\left(\frac{\sqrt{4mk - \gamma^2}}{2m} t\right)$

$$r_1 = -\frac{\gamma}{2m} + \frac{\sqrt{4mk - \gamma^2}}{2m} i$$

$$r_2 = -\frac{\gamma}{2m} - \frac{\sqrt{4mk - \gamma^2}}{2m} i$$

$$\frac{\sqrt{4mk - \gamma^2}}{2m}$$

To study the effects of damping, let's consider $m=1$, $k=1$, and $\gamma=c$ variable. Then, we arrive to

$$u'' + c u' + u = 0$$

Show solns. depending on c in MAPLE!

Specific Solns. for Damped Vibrations without external force

$$mu'' + cu' + ku = 0, \quad m, c, k > 0.$$

If $m=1, k=1$, then

$$\text{IVP} \quad \begin{cases} u'' + cu' + u = 0 \\ u(0) = 1, u'(0) = 1 \end{cases}$$

General Soln:

$$u(t) = C_1 e^{\left(\frac{-c + \sqrt{c^2 - 4}}{2}\right)t} + C_2 e^{\left(\frac{-c - \sqrt{c^2 - 4}}{2}\right)t}$$

Overdamped: $c=3$

$$u(t) = \left(\frac{1+\sqrt{5}}{2}\right) e^{\frac{-3+\sqrt{5}}{2}t} + \left(\frac{1-\sqrt{5}}{2}\right) e^{\frac{-3-\sqrt{5}}{2}t}$$

Critically damped: $c=2$

$$u(t) = e^{-t} + 2te^{-t}$$

Damped: $c=1/10$

$$u(t) = \frac{\sqrt{399}}{19} e^{-\frac{t}{20}} \sin\left(\frac{\sqrt{399}}{20}t\right) + e^{-\frac{t}{20}} \cos\left(\frac{\sqrt{399}}{20}t\right)$$

Undamped Free Vibrations. (no external force)

In this case $\gamma = 0$ { ideal case because there is always damping }

Equation (5.5) reduces to

$$m u'' + k u = 0 \quad \text{or} \quad u'' + \frac{k}{m} u = 0$$

Charact. polyn.: $r^2 + \frac{k}{m} = 0 \Rightarrow r = \pm i \left(\frac{k}{m} \right)^{1/2}$

and general Soln.:

$$u(t) = A \cos \left(\sqrt{\frac{k}{m}} t \right) + B \sin \left(\sqrt{\frac{k}{m}} t \right) \quad (5.6)$$

Usually a new constant

$$\omega_0^2 = k/m \Rightarrow \boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

which is called natural frequency of vibration is introduced and (5.6) is written as

$$\boxed{u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)} \quad (5.7)$$

A more useful representation is discussed in the next page.

$$m u'' + k u = 0$$

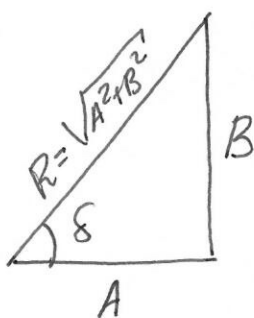
$$r^2 + \frac{k}{m} = 0, \quad r = \pm i \sqrt{\frac{k}{m}}$$

$$\omega_0^2 = \frac{k}{m} \quad \text{Nat. freq. of vibration.}$$

$$u(t) = A \cos\left(\sqrt{\frac{k}{m}} t\right) + B \sin\left(\sqrt{\frac{k}{m}} t\right)$$

$$= A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

To obtain a more useful representation introduce an angle " δ ".



$$\tan \delta = \frac{B}{A}$$

$$\sin \delta = \frac{B}{R}$$

$$\cos \delta = \frac{A}{R}$$

$$\Rightarrow B = R \sin \delta$$

$$A = R \cos \delta.$$

$$\Rightarrow u(t) = R \cos \delta \cos(\omega_0 t) + R \sin \delta \sin(\omega_0 t)$$

$$= R \cos(\omega_0 t - \delta)$$

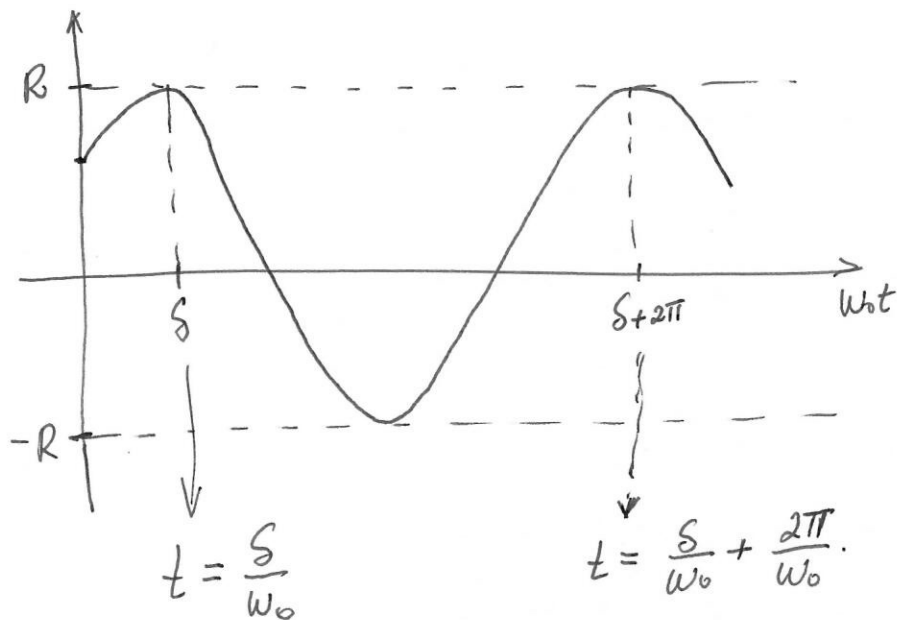
$\downarrow$ amplitude of motion $\swarrow$ phase angle

Natural freq. of vibration.

$$\text{period} = \frac{2\pi}{\omega_0} = T$$

Graph. of

$$u(t) = R \cos(\omega_0 t - \delta).$$



This behavior is harder to predict from the original form of the solution :

$$u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

Computation of R and S from A and B .

$$R = \sqrt{A^2 + B^2}$$

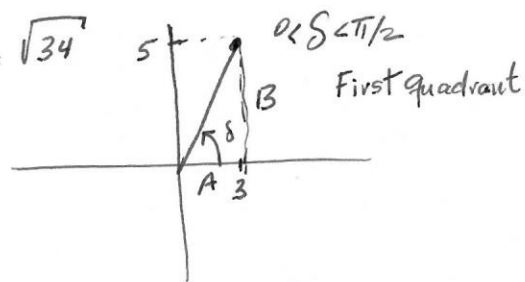
Also $\tan S = \frac{B}{A}$

To find S needs to consider the signs of A and B to identify the tangent branch.

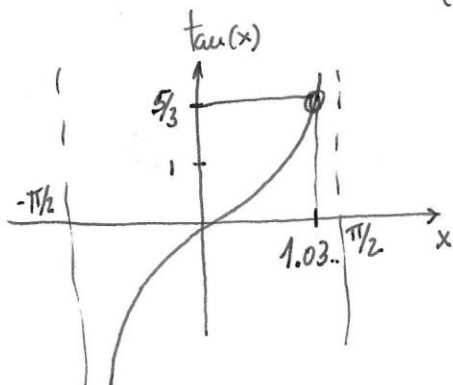
Examples: (I) $U(t) = 3 \cos(2t) + 5 \sin(2t)$

$$A = 3, \quad B = 5 \Rightarrow R = \sqrt{9 + 25} = \sqrt{34}$$

$$\tan S = \frac{B}{A} = \frac{5}{3}$$



$$\Rightarrow S = \arctan(5/3) \approx 1.03\dots$$



$$\Rightarrow U(t) = \sqrt{34} \cos(2t - 1.03)$$

Similar computation any time $A > 0, B > 0$.

$$S = \arctan\left(\frac{B}{A}\right).$$

(II)

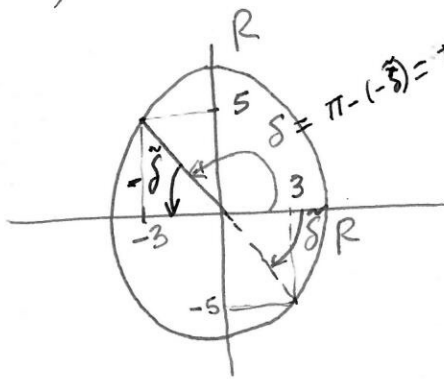
If $U(t) = -3\cos(2t) + 5\sin(2t)$

$$\tan(\delta) = \frac{5}{-3}$$

$$\Rightarrow \tilde{\delta} = \arctan\left(\frac{5}{-3}\right) = \arctan\left(-\frac{5}{3}\right) \approx -1.03$$

Using Scient. Calculator.

Now,

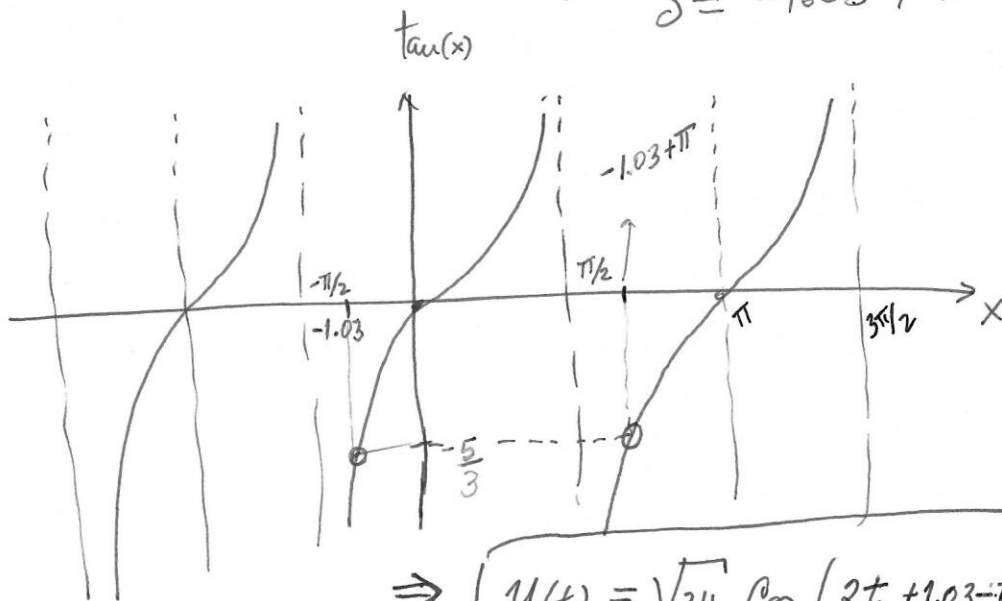


$$\delta = \pi - (-\tilde{\delta}) = \pi + \tilde{\delta} \quad \frac{\pi}{2} < \delta < \pi \quad 2^{\text{nd}} \text{ Quadrant}$$

$$-\frac{\pi}{2} < \tilde{\delta} < 0 \quad 4^{\text{th}} \text{ Quadrant.}$$

$$\tilde{\delta} < 0 \Rightarrow \boxed{\delta = \tilde{\delta} + \pi}$$

$$\Rightarrow \delta = -1.03 + \pi \quad (\text{in } 2^{\text{nd}} \text{ quad.})$$



$$\Rightarrow \boxed{U(t) = \sqrt{34} \cos(2t + 1.03 - \pi)}$$

So $A < 0, B > 0$

$$\Rightarrow \delta = \pi + \arctan\left(\frac{B}{A}\right)$$

III

If $u(t) = -3 \cos 2t - 5 \sin 2t$ $A < 0, B < 0$.

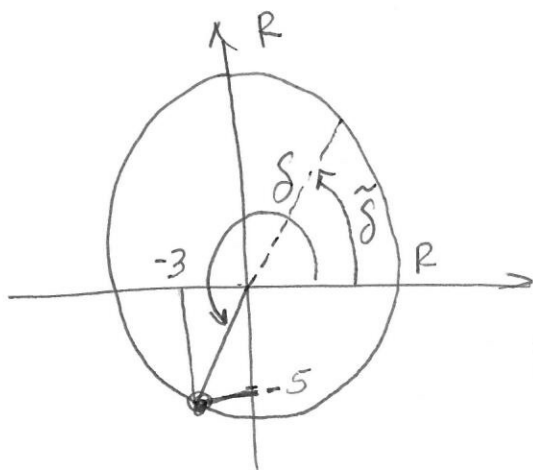
$$\tan \delta = \frac{-5}{-3} = \frac{5}{3} \Rightarrow \tilde{\delta} = \arctan\left(\frac{5}{3}\right) \approx 1.03$$

Using Scientific Calculator

$$0 < \tilde{\delta} < \pi/2$$

$$\pi < \delta < 3\pi/2$$

3rd Quadrant.



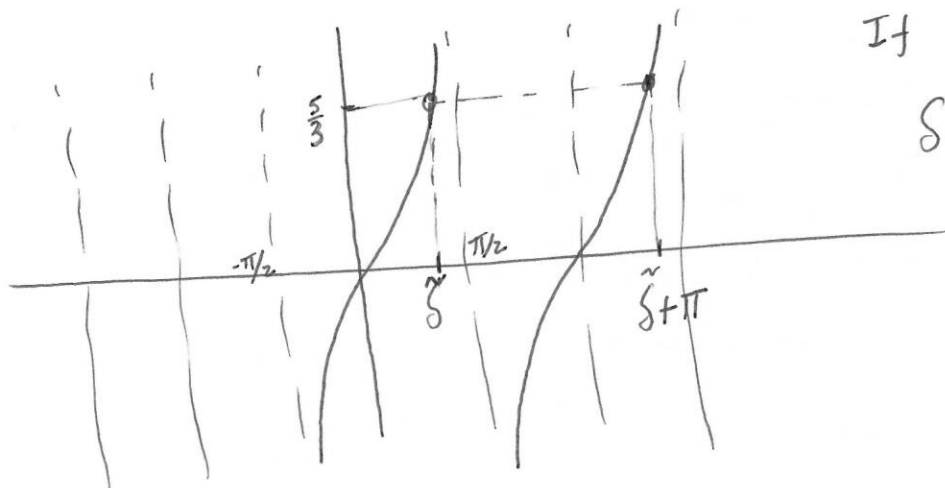
$$0 < \tilde{\delta} < \pi/2$$

1st Quadrant

$$\boxed{\delta = \pi + \tilde{\delta}} \Rightarrow u(t) = \sqrt{34} \cos(2t - 1.03 - \pi)$$

If $A < 0, B < 0$

$$\delta = \pi + \arctan\left(\frac{B}{A}\right)$$



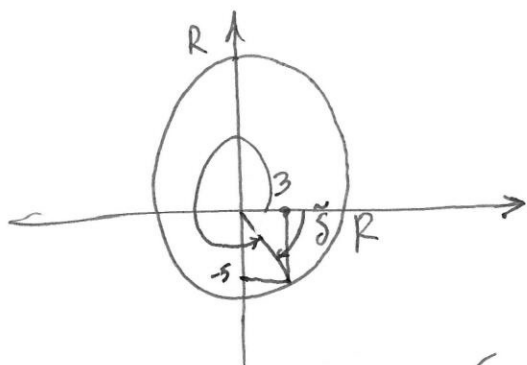
IV

If $u(t) = 3 \cos(2t) - 5 \sin(2t)$, $A > 0, B < 0$
4th Quadrant

$$\tan \delta = \left(-\frac{5}{3}\right) \Rightarrow \tilde{\delta} = \arctan\left(-\frac{5}{3}\right) \approx -1.03$$

$$-\frac{\pi}{2} < \tilde{\delta} < 0$$

$$\Rightarrow \delta = \tilde{\delta} + 2\pi \Rightarrow u(t) = \sqrt{34} \cos(2t + 1.03)$$



However,

$$\begin{aligned} \cos(\omega t + \delta) &= \cos(\omega t - \tilde{\delta} - 2\pi) = \\ &= \cos(\omega t - \tilde{\delta}) \end{aligned}$$

So it's ok to use $\tilde{\delta}$ instead of $\delta = \tilde{\delta} + 2\pi$

Summarizing

If $u(t) = A \cos(\omega t) + B \sin(\omega t)$

$$\Rightarrow u(t) = R \cos(\omega t - \delta)$$

Where $R = \sqrt{A^2 + B^2}$

and

$$\delta = \begin{cases} \arctan(B/A), & \text{if } A \geq 0, B \geq 0. \text{ I Quad.} \\ \arctan(B/A) + \pi, & \text{if } A \leq 0, B \geq 0. \text{ II Quad.} \\ \arctan(B/A) - \pi, & \text{if } A \leq 0, B \leq 0. \text{ III Quad.} \\ \arctan(B/A), & \text{if } A \geq 0, B \leq 0. \text{ IV Quad.} \end{cases}$$