3.9 Forced Vibrations.

I) Forced Vibrations with Damping.

$$\boxed{mu'' + \gamma u' + ku = F_0 \cos \omega t} \quad (1.1)$$

$F_0 \cos \omega t \rightarrow$ Typical External force.

General Soln:

a) First find a fundamental set of Solns. for corresponding homogeneous equation:

$$\boxed{mu'' + \gamma u' + ku = 0} \quad (1.2)$$

Charact. polyn: $mr^2 + \gamma r + k = 0$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

Three possibilities for r_1 and r_2

i) r_1, r_2 real and $r_1 \neq r_2$

$$u_1 = e^{r_1 t}, \quad u_2 = e^{r_2 t}$$

ii) $r_1 = r_2$ real $\Rightarrow u_1 = e^{r_1 t}, \; u_2 = te^{r_1 t}$

iii) $r_1 = \alpha + i\beta, \; r_2 = \alpha - i\beta, \; \beta \neq 0$

$$u_1 = e^{\alpha t} \cos \beta t, \quad u_2 = e^{\alpha t} \sin \beta t.$$

In all those cases:

If real, $r_1 < 0$, $r_2 < 0$ why?

If complex, $\alpha < 0$, why?

If $u_p(t)$ is a particular soln. of (1.1), then its general soln:

$$\boxed{u(t) = \underbrace{C_1 u_1(t) + C_2 u_2(t)}_{\text{general soln. of homog. equ.}} + u_p(t) = u_h(t) + u_p(t)}$$

From the previous consideration of the roots r_1 and r_2

$$u_h(t) \xrightarrow[t \to \infty]{} 0$$

then,
$$u(t) = u_h(t) + u_p(t) \xrightarrow[t \to \infty]{} u_p(t)$$

$u_h(t)$ is called the transient soln. and

$u_p(t)$ is called the steady or forced response soln.

$u_h(t)$ depends on the ICs. The energy provided by them to the system is dissipated by the damping force.

For the particular form of the forcing function in (1.1)

$$F_0 \cos(\omega t),$$

it holds that

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

or $\quad u_p(t) = R\cos(\omega t - \delta).$

where $\boxed{R = \dfrac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2) + \gamma^2\omega^2}}}$

Notice that when ω is close to ω_0 and γ is small, the amplitude R is large even for small F_0. This phenomenon is known as resonance.

Go to MAPLE

$$\begin{cases} u'' + 2bu' + \omega_0^2 u = F\cos(\omega t) \\ u(0)=1, \quad u'(0)=1 \end{cases}$$

i) Small b and $\omega < \omega_0$, or weak damping

ii) larger b and $\omega < \omega_0$, or strong damping

iii) Small b and $\omega = \omega_0$ Resonance.

II) Forced Vibrations without Damping.

$$\boxed{u'' + w_0^2 u = F\cos(wt)} \qquad (4.1)$$

Two Cases:

a) $\boxed{w \neq w_0}$

Charact. polyn: $r^2 + w_0^2 = 0 \Rightarrow r_{1,2} = \pm i\sqrt{w_0^2} = \pm i w_0$

then $\quad u_h(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t)$. (4.2)

Also $\quad u_p(t) = A\cos(wt) + B\sin(wt) \qquad (4.3)$

To determine A and B, we substitute (4.3) into (4.1).

$$u_p' = -Aw\sin wt + Bw\cos wt$$

$$u_p'' = -Aw^2\cos wt - Bw^2\sin wt$$

$$w_0^2\left(u_p = A\cos wt + B\sin wt\right)$$

Then $\quad u_p'' + w_0^2 u_p = A(w_0^2 - w^2)\cos wt + B(w_0^2 - w^2)\sin wt$

$$= F\cos wt$$

Thus,

$$A(w_0^2 - w^2) = F$$

$$B(w_0^2 - w^2) = 0 \Rightarrow B = 0$$

$$\Rightarrow \boxed{A = \frac{F}{w_0^2 - w^2}} \qquad (4.2)$$

Therefore, general Soln is

$$\boxed{u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F}{\omega_0^2 - \omega^2} \cos \omega t} \quad (5.1)$$

C_1 and C_2 are determined using ICs.

after substituting (5.1) (homog. + particular solns) into (4.1).

Obviously, for ω close to ω_0 the solution becomes very large.

b) $\boxed{\omega = \omega_0}$

$$\boxed{\begin{aligned} u(t) = {} & C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \\ & + A t \cos(\omega_0 t) + B t \sin(\omega_0 t) \end{aligned}} \quad (5.2)$$

Substitution into (4.1) leads to

$$\boxed{u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{2\omega_0} t \sin \omega_0 t} \quad (5.3).$$

Show this!

A Beat Soln:

Consider
$$\begin{cases} u'' + \omega_0^2 u = F \cos \omega t \\ u(0) = 0, \quad u'(0) = 0 \end{cases}$$

The general soln. is given by (5.1)

$$u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F}{\omega_0^2 - \omega^2} \cos \omega t$$

Using ICs:
$$C_1 = \frac{-F}{(\omega_0^2 - \omega^2)}, \quad C_2 = 0$$

Then,
$$u(t) = \frac{F}{(\omega_0^2 - \omega^2)} \left(\cos \omega t - \cos \omega_0 t \right)$$

If $A = \dfrac{(\omega_0 + \omega)t}{2}$, $B = \dfrac{(\omega_0 - \omega)t}{2}$

then
$$\cos(\omega t) = \cos(A - B) = \cos A \cos B + \sin A \sin B.$$
$$\cos(\omega_0 t) = \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\Rightarrow \cos(\omega t) - \cos(\omega_0 t) = 2 \sin A \sin B = 2 \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right)$$

Thus,
$$\boxed{u(t) = \left[\frac{2F}{\omega_0^2 - \omega^2} \sin\left(\frac{\omega_0 - \omega}{2} t\right) \right] \sin\left(\frac{\omega_0 + \omega}{2} t\right)}$$