Previous to exercise 15 Sect 4.2

$$y'' + y = 0 \qquad (0.1)$$

$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$ Complex Conjugate roots.

Fund. set of Solns:

$$y_1(t) = \sin t, \quad y_2(t) = \cos t$$

We will see that for higher order homog. constant Coefficient equs.

If there is a complex root its conjugate is also a root.

In fact, if $\quad r_1 = \alpha + i\beta$

then $\quad r_2 = \alpha - i\beta$

and there are two Solns. linearly indep. $(\omega \neq 0)$

$$y_1(t) = e^{\alpha t}\cos(\beta t), \quad y_2(t) = e^{\alpha t}\sin(\beta t).$$

So general Soln. of (0.1)

$$y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t).$$

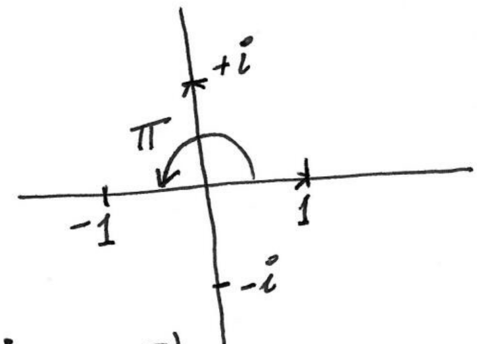
4.2 Homogeneous Equations with Constant Coefficients

Exercise #15 Find the general solution of

$$y^{(6)} + y = 0$$

Answ: $r^6 + 1 = 0 \Rightarrow r^6 = -1$

Complex plane:

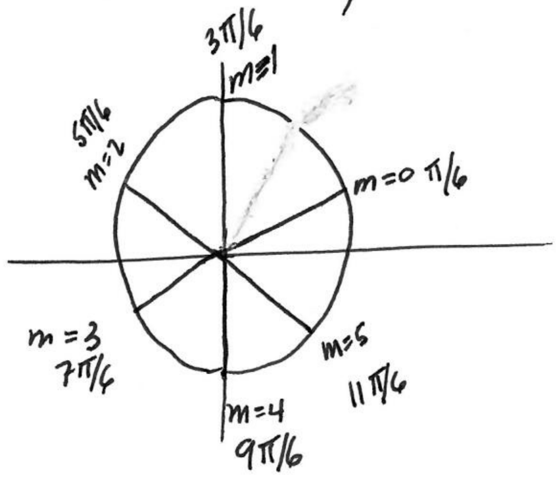


$$-1 = e^{i\pi}$$

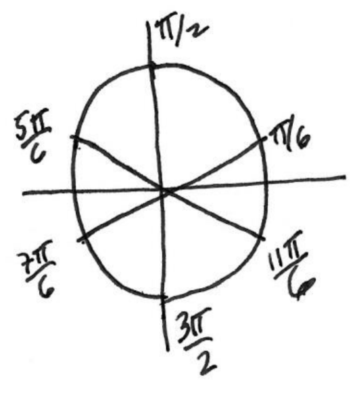
but also, $-1 = e^{i(\pi + 2m\pi)}, \quad m = 0, 1, 2 \ldots$ Counterclockwise

Therefore, $(-1)^{1/6} = \left[e^{i(\pi + 2m\pi)}\right]^{1/6} = e^{i(\pi/6 + 2m\frac{\pi}{6})}$

$$= e^{i(\pi/6 + m\pi/3)}, \quad m = 0, 1, .., 5$$



More precise



These roots can be written in terms of trig. functions and from these representation we can obtain the corresponding solutions:

$m=0 \quad e^{i\pi/6} = \cos(\pi/6) + i\sin(\pi/6)$

$= \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$y_1(t) = e^{\sqrt{3}/2\, t}\cos(t/2), \quad y_2(t) = e^{\sqrt{3}/2\, t}\sin(t/2).$

$m=1 \quad e^{i\pi/2} = \cos(\underset{=0}{\pi/2}) + i\sin(\underset{=1}{\pi/2}) = i$

$y_3(t) = \sin t, \quad y_4(t) = \cos t$

$m=2 \quad e^{i\frac{5\pi}{6}} = \cos\left(\frac{5\pi}{6}\right) + i\sin(5\pi/6) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$

$y_5(t) = e^{-\sqrt{3}/2\, t}\cos(t/2), \quad y_6(t) = e^{\frac{\sqrt{3}}{2}t}\sin(t/2)$

$m=3 \quad e^{i7\pi/6} = \cos(7\pi/6) + i\sin(7\pi/6) = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$

$y_7(t) = e^{-\sqrt{3}/2\, t}\cos(-t/2) = y_5(t),$

$y_8(t) = e^{-\sqrt{3}/2\, t}\sin(-t/2) = -e^{-\sqrt{3}/2\, t}\sin(t/2) = -y_6(t)$

$m=4 \quad e^{i\frac{3\pi}{2}} = \cos\overset{=0}{(3\pi/2)} + i\sin\overset{=1}{(3\pi/2)} = -1i = -i$

Complex Conjugate to the case $m=1$

Same Solns: $y_3(t) = \sin t, \quad y_4(t) = \cos t.$

$m=5 \quad e^{i\,11\pi/6} = \cos\left(\frac{11\pi}{6}\right) + i\sin(11\pi/6)$

$= \frac{\sqrt{3}}{2} - i\frac{1}{2}$

Complex conjugate of roots $m=0$ case.

Same solns:

$y_1(t) = e^{\sqrt{3}/2\, t}\cos(t/2), \quad y_2(t) = e^{\sqrt{3}/2\, t}\sin(t/2).$

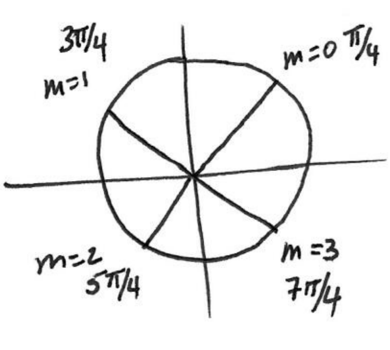
So general Soln:

$$\boxed{\begin{aligned} y(t) = {} & C_1 e^{\sqrt{3}/2\, t}\cos(t/2) + C_2 e^{\sqrt{3}/2\, t}\sin(t/2) \\ & + C_3\sin t + C_4\cos t + C_5 e^{-\sqrt{3}/2\, t}\cos(t/2) + C_6 e^{-\sqrt{3}/2\, t}\sin(t/2) \end{aligned}} \qquad (H3.1)$$

Try: $y^{(4)} + 4y = 0 \qquad r^4 + 4 = 0 \Rightarrow r^4 = -4 = (4e^{i\pi})$

or $r = (4e^{i\pi})^{1/4} = (4)^{1/4} e^{i\left(\frac{\pi}{4} + \frac{2m\pi}{4}\right)}$

or $r = e^{i(\pi/4 + m\pi/2)}$



Exercise 22

$$y^{(4)} + 2y'' + y = 0$$

$$r^4 + 2r^2 + 1 = 0$$

possible roots : $r = \pm 1$, but none of them is a root.

Try factorization:

$$0 = r^4 + 2r^2 + 1 = (r^2 + 1)^2$$

Then the roots of $r^2 = -1 \Rightarrow r = \pm i$

are repeated twice:

and $\boxed{y(t) = C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t}$

(H4.1)

4.3 Nonhomogeneous Equ. of Constant Coefficient.

Find form of general Soln of

Example.

$$y^{(6)} + y = 3\, e^{-\sqrt{3}/2\, t} \sin(t/2) + 5t^2$$

$$y(t) = \underset{(H3.1)}{y_h(t)} + y_p(t)$$

or $$y(t) = y_h(t) + At\, e^{-\sqrt{3}/2\, t} \sin(t/2) + Bt\, e^{-\sqrt{3}/2\, t} \cos(t/2)$$

$$+ \; C + Dt + Et^2$$

Example Nonhomogeneous

$$y^{(4)} + 2y'' + y = -7t \cos t$$

Find form of general Soln.:

$$y(t) = \underset{(H4.1)}{y_h(t)} + y_p(t)$$

$$y(t) = y_h(t) + At^2 \cos t + Bt^2 \sin t$$

Sect. 4.3 Problems

#6) $y^{(4)} + 2y'' + y = 3 + \cos 2t$

$r^4 + 2r^2 + 1 = 0 \Leftrightarrow (r^2+1)^2 = 0$

$r_1 = i$, $r_2 = -i$ with multiplicity two

thus, $y_h(t) = C_1 \cos t + C_2 \sin t + C_3 t \sin t + C_4 t \cos t$.

and $y_p(t) = A_0 + A_1 \cos 2t + A_2 \sin 2t$.

#8) $y^{(4)} + y^{(3)} = \sin 2t$

$r^4 + r^3 = 0 \Leftrightarrow r^3(r+1) = 0 \Rightarrow \underset{m:3}{r_1 = 0}, \; r_2 = -1$

$\Rightarrow y_h(t) = C_0 + C_1 t + C_2 t^2 + C_3 e^{-t}$

$y_p(t) = A \sin 2t + B \sin 2t$

If $y^{(4)} + y^{(3)} = 3t$

then, $y_p(t) = t^3 [A_0 + A_1 t]$

If $y^{(4)} + y^{(3)} = t^2 \sin 2t$

$y_p(t) = (A_0 + A_1 t + A_2 t^2) \sin 2t + (B_0 + B_1 t + B_2 t^2) \cos 2t$.