4.4 Method of Variation of parameters. Higher order equations.

Consider a third order equation.

$$\boxed{y'''(t) + p_1 y''(t) + p_2 y'(t) + p_3 y(t) = g(t)} \quad (1)$$

and Suppose $\{y_1, y_2, y_3\}$ is a fundamental set of Solutions of (1)

MVP Consists of finding a particular solution of (1) as

$$\boxed{y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t) + u_3(t)\, y_3(t)} \quad (2)$$

What do you need to do?

$$y_p' = \underbrace{u_1' y_1 + u_2' y_2 + u_3' y_3} + u_1 y_1' + u_2 y_2' + u_3 y_3'$$

Before obtaining y_p'' and y_p''', we Simplify y_p' imposing the condition

$$\boxed{u_1' y_1 + u_2' y_2 + u_3' y_3 = 0}$$

then

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_3 y_3'' + \underbrace{u_1' y_1' + u_2' y_2' + u_3' y_3'}$$

Before obtaining y_p''', we simplify y_p'' by imposing the condition

$$\boxed{u_1' y_1' + u_2' y_2' + u_3' y_3' = 0}$$

then,
$$y_p''' = u_1' y_1'' + u_2' y_2'' + u_3' y_3'' + u_1 y_1''' + u_2 y_2''' + u_3 y_3'''$$

Subst. into (1) $\quad y_p''' + p_1 y_p'' + p_2 y_p' + p_3 y_p = g(t)$

$$u_1 y_1''' + u_2 y_2''' + u_3 y_3''' + u_1' y_1'' + u_2' y_2'' + u_3' y_3'' +$$

$$+ p_1 u_1 y_1'' + p_1 u_2 y_2'' + p_1 u_3 y_3'' +$$

$$+ p_2 u_1 y_1' + p_2 u_2 y_2' + p_2 u_3 y_3' +$$

$$+ p_3 u_1 y_1 + p_3 u_2 y_2 + p_3 u_3 y_3 \overset{\text{should be}}{=\!=} g(t).$$

$$u_1 \overbrace{\left(y_1''' + p_1 y_1'' + p_2 y_1' + p_3 y_1\right)}^{0} +$$

$$+ u_2 \overbrace{\left(y_2''' + p_1 y_2'' + p_2 y_2' + p_3 y_2\right)}^{0} +$$

$$+ u_3 \overbrace{\left(y_3''' + p_1 y_3'' + p_2 y_3' + p_3 y_3\right)}^{0} +$$

$$+ \boxed{u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = g(t)}$$

Therefore, we are let with the following linear System of equations for u_1', u_2', u_3' at any point t.

$$\begin{cases} y_1 u_1' + y_2 u_2' + y_3 u_3' = 0 \\ y_1' u_1' + y_2' u_2' + y_3' u_3' = 0 \\ y_1'' u_1' + y_2'' u_2' + y_3'' u_3' = g(t) \end{cases}$$

Cramer's rule:

$$\Delta = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{vmatrix}$$

$$= W[y_1, y_2, y_3](t)$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ g & y_2'' & y_3'' \end{vmatrix}}{W(t)} = \left(\frac{y_2 y_3' - y_3 y_2'}{W(t)}\right) g(t) = \frac{W_1(t)}{W(t)} g(t)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & g & y_3'' \end{vmatrix}}{W(t)} = \frac{(-1)\, y_1 y_3' - y_3 y_1'}{W(t)} g(t) = \frac{+W_2(t)}{W(t)} g(t)$$

$$u_3' = \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & g \end{vmatrix}}{W(t)} = \frac{y_1 y_2' - y_2 y_1'}{W(t)} g(t) = \frac{W_3(t)}{W(t)} g(t)$$

4.4 #4) $y''' + y' = \sec t, \quad -\pi/2 < t < \pi/2.$

$r^3 + r = 0 \Rightarrow r(r^2+1) = 0 \Rightarrow r_1 = 0, \quad r_2 = \pm i$

Solns. of homogeneous equ.

$y_1(t) = 1, \quad y_2(t) = \cos t, \quad y_3(t) = \sin t$ They form a fund. set.

<u>Particular Soln.:</u>

$$y_p(t) = u_1(t)\, 1 + u_2(t) \cos t + u_3(t) \sin t.$$

Using the method of Variation of parameters we need to solve the following system for u_1', u_2', and u_3'.

$$\begin{cases} u_1' + \cos t\, u_2' + \sin t\, u_3' = 0 \\ 0 - \sin t\, u_2' + \cos t\, u_3' = 0 \\ 0 - \cos t\, u_2' - \sin t\, u_3' = \sec t \end{cases}$$

Using Kramer's rule:

$$u_1' = \frac{\begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \sec t & -\cos t & -\sin t \end{vmatrix}}{\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix}} = \frac{\sec t \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}}{1} = \sec t.$$

$$\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = W(y_1, y_2, y_3)$$

So

$$\boxed{u_1'(t) = \sec t} \Rightarrow \boxed{u_1(t) = \int \sec t \, dt = \ln\left(|\sec t + \tan t|\right)}$$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 & -\sin t \\ 0 & 0 & \cos t \\ 0 & \sec t & -\sin t \end{vmatrix}}{W(y_1, y_2, y_3) = 1} = \frac{\begin{vmatrix} 0 & \cos t \\ \sec t & -\sin t \end{vmatrix}}{1} = -\sec t \cos t$$

or

$$u_2' = -1 \Rightarrow \boxed{u_2(t) = -t}$$

and

$$u_3' = \frac{\begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \sec t \end{vmatrix}}{1} = \frac{\begin{vmatrix} -\sin t & 0 \\ -\cos t & \sec t \end{vmatrix}}{1} = -\frac{\sin t}{\cos t}$$

or

$$u_3' = -\tan t \Rightarrow u_3(t) = -\int \tan t \, dt = \ln|\sec t|$$

$$\boxed{u_3(t) = \ln|\cos t|}$$

Therefore,

$$\boxed{y_p(t) = \ln\left(|\sec t + \tan t|\right) - t\cos t + \ln\left(|\cos t|\right)\sin t}$$

and the general soln.:

$$\boxed{y(t) = C_1 + C_2\cos t + C_3 \sin t + y_p(t)}$$