

5.1 Review of Power Series.

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Def. - A power series is an infinite series of the form:

$$S(x) = \sum_{n=1}^{\infty} a_n (x-x_0)^n \quad (1)$$

(1) always converge for $x = x_0$.

$$S(x_0) = 0.$$

Def. - $S(x)$ is abs. convergent if

$$\sum_{n=1}^{\infty} |a_n (x-x_0)^n| = \sum_{n=1}^{\infty} |a_n| |x-x_0|^n \text{ converges.}$$

Def. - There is a nonnegative number ρ , called radius of convergence, such that

$$\sum_{n=1}^{\infty} a_n (x-x_0)^n \text{ Converges absolutely for } |x-x_0| < \rho$$

and diverges for $|x-x_0| > \rho$.

Remarks: - If $S(x)$ only converges at $x = x_0$, then $\rho = 0$

- At x such that $|x-x_0| = \rho$, the series may or may not converge.

Thm

If the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

is absolutely convergent,
then it is convergent.

The converse is not true. It means the infinite series may converge but its absolute value not.

Example: $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges (alternating series test)

However, $S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges.

This is called the harmonic series.

Ratio Test Consider $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

If $a_n \neq 0$ and $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$,
then for a fixed value of x

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = \lim_{n \rightarrow \infty} |x-x_0| \left| \frac{a_{n+1}}{a_n} \right|$$

$$= |x-x_0| L$$

Therefore at that value of x ,

- 1) the series converges absolutely if $|x-x_0| L < 1$
- 2) " " diverges if $|x-x_0| L > 1$.
- 3) The test is inconclusive if $|x-x_0| L = 1$.

Example 2. (book)

Determine radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n 2^n}$$

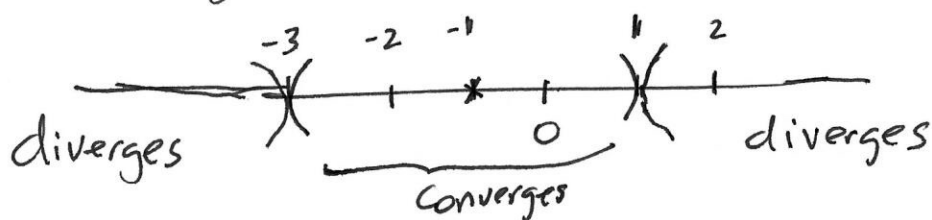
Applying ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x+1)^{n+1}}{(n+1) 2^{n+1}}}{\frac{(x+1)^n}{n 2^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1) n}{2(n+1)} \right|$$

$$= \frac{|x+1|}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{|x+1|}{2}$$

So the P. Series Converges if

$$\frac{|x+1|}{2} < 1 \Rightarrow |x+1| < 2 \Rightarrow -3 < x < 1$$



at $\boxed{x = -3}$

$$\begin{aligned} S(-3) &= \sum_{n=1}^{\infty} \frac{(-3+1)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{-2}{2}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ Converges.} \end{aligned}$$

However, at $\boxed{x=1}$

$$S(1) = \sum_{n=1}^{\infty} \frac{2^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

So interval of convergence is given by

$$[-3, 1). \text{ and } \rho = 2.$$

$$\text{If } \sum_{n=0}^{\infty} a_n (x-x_0)^n \rightarrow f(x).$$

$$\text{or } f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ and } |x-x_0| < \rho, \rho > 0. \\ \text{(interval of convergence)}$$

then f is conts. and has derivatives of all orders.
for $|x-x_0| < \rho$.

Moreover,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

⋮

differentiation
term by
term is
valid.

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$$\sum_{m=2}^{\infty} m(m-1) a_m X^{m-2} + X \underbrace{\sum_{k=1}^{\infty} k a_k X^{k-1}}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} X^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} X^n$$

How

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$$\sum_{n=2}^{\infty} n(n-1) a_n X^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n X^n$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} X^m - \sum_{n=0}^{\infty} n(n-1) a_n X^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n(n-1) a_n] X^n$$