5.4 Euler Equations.

Consider

$$x^2 y'' + \alpha x y' + \beta y = 0, \quad x > 0. \quad (1.1)$$

This equation has a singular point at $x_0 = 0$.
So a solution in the form of a power series at $x_0 = 0$ may not exists.

Consider our previous experience with Constant coefficients homogeneous linear Second order equations

$$ay'' + by' + cy = 0$$

where we look for solutions as

$$y(x) = e^{rx}$$

And we arrive to a characteristic polyn.

$$ar^2 + br + cr = 0$$

Then, we consider the three cases:

i) $r_1 \neq r_2$ real and different

ii) $r_1 = r_2$ real repeated.

iii) $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$.

Also, consider that

$$\frac{d}{dx}(x^r) = \frac{d}{dx}\left((e^{\ln x})^r\right) = \frac{d}{dx}\left(e^{r\ln x}\right)$$

$$= r(\ln x)' e^{r\ln x} = \frac{r}{x} e^{r\ln x} = \frac{r}{x} x^r$$

$$= r\, x^{r-1}.$$

and $\frac{d^2}{dx^2}(x^r) = \frac{d}{dx}\left(r x^{r-1}\right) = r(r-1)x^{r-2}.$

Therefore, if we propose a soln. for (1.1)

as $\qquad y(x) = x^r$

we will arrive to

$$0 = x^2 y'' + \alpha x y' + \beta y$$

$$= x^2\left(r(r-1)x^{r-2}\right) + \alpha x\left(r x^{r-1}\right) + \beta x^r$$

$$= \left[r(r-1) + \alpha r + \beta\right] x^r$$

Therefore, the equation (1.1) is Satisfied by $y(x) = x^r$ if

$$r(r-1) + \alpha r + \beta = 0$$

or

$$r^2 + (\alpha-1)r + \beta = 0$$

or

$$r_{1,2} = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

Three cases arise:

i) $(\alpha-1)^2 - 4\beta > 0 \Rightarrow r_1 \neq r_2$ real.

ii) $(\alpha-1)^2 = 4\beta \Rightarrow r_1 = r_2$ real.

iii) $(\alpha-1)^2 - 4\beta < 0 \Rightarrow r_1 = \lambda + i\mu$
$r_2 = \lambda - i\mu$.

<u>Case (i)</u> There are two solutions in the form $y(x) = x^r$.

$$y_1(x) = x^{r_1} \quad \text{and} \quad y_2(x) = x^{r_2}$$

It can be easily proved that they are linearly indep. or that they form a fund. set by considering their Wronskian.

$$W(y_1,y_2)(x) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1-1} & r_2 x^{r_2-1} \end{vmatrix} = (r_2-r_1)\,x^{r_1+r_2-1} \neq 0$$

Since $r_1 \neq r_2$.

As a consequence, the general soln. can be written as

$$\boxed{y(x) = C_1 x^{r_1} + C_2 x^{r_2}.}$$

<u>Case (ii)</u> <u>Repeated roots</u> <u>$r_1 = r_2$</u>

One Solution: $y_1(x) = x^{r_1}$,

Second Solution can be obtained applying reduction order: $y_2(x) = \ln(x)\, x^{r_1}$

It can be easily prove that they form a fund. set

Therefore, general Soln can be written as

$$\boxed{y(x) = C_1 x^{r_1} + C_2 \ln(x)\, x^{r_1}.}$$

Case (iii) Complex Conjugate roots

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu.$$

Two complex Solns:

$$z_1(x) = x^{r_1} = x^{\lambda + i\mu} = e^{\ln x(\lambda + i\mu)}$$

$$= e^{\lambda \ln x} \cdot e^{i\mu \ln x}$$

$$= e^{\lambda \ln x}\left(\cos(\mu \ln x) + i \sin(\mu \ln x)\right) \quad (5.1)$$

Similarly,

$$z_2(x) = e^{\lambda \ln x}\left(\cos(\mu \ln x) - i \sin(\mu \ln x)\right) \quad (5.2)$$

Two real Solns y_1 and y_2 can be obtained from complex linear combination of $\underset{z_1}{(5.1)}$ and $\underset{z_2}{(5.2)}$.

They are

$$y_1(x) = e^{\lambda \ln x} \cos(\mu \ln x), \quad y_2(x) = e^{\lambda \ln x} \sin(\mu \ln x)$$

Computing their wronskian is possible to prove that they are linearly indep. Therefore,

GENERAL Soln:

$$\boxed{y(x) = C_1 e^{\lambda \ln x} \cos(\mu \ln x) + C_2 e^{\lambda \ln x} \sin(\mu \ln x)}$$

or

$$\boxed{y(x) = C_1 x^{\lambda} \cos(\mu \ln x) + C_2 x^{\lambda} \sin(\mu \ln x).}$$

Euler Equation

Some problems: $\boxed{x^2 y'' - \alpha x y' + \beta y = 0}$

Exercise #12) $x^2 y'' - 4xy' + 4y = 0, \quad x > 0.$

$\alpha = -4, \; \beta = 4.$

The Characteristic eqn. is

$r^2 + (\alpha - 1) r + \beta = 0$ or $r^2 + (-5) r + 4 = 0$

$$r_{1,2} = \frac{5 \pm \sqrt{25-16}}{2} = \begin{cases} 8/2 = 4 \\ 2/2 = 1 \end{cases}$$

General soln.

Therefore, $\boxed{y(x) = C_1 x^4 + C_2 x.}$

Ex. 9) $x^2 y'' - 5xy' + 9y = 0, \quad x > 0, \quad \alpha = -5, \; \beta = 9$

Ch. eqn: $r^2 + (-6) r + 9 = 0$

$$r_{1,2} = \frac{6 \pm \sqrt{36-36}}{2} = \frac{6}{2} = 3 \text{ only root}$$

So $y_1(x) = x^3, \quad y_2(x) = \ln(x)\, x^3$

Gener. Soln. $\boxed{y(x) = C_1 x^3 + C_2 \ln(x)\, x^3.}$

Ex. 14) $4x^2y'' + 8xy' + 17y = 0, \quad x>0$

Find a soln as : $y(x) = x^r$

$$4r(r-1)x^r + 8rx^r + 17x^r = 0$$

$$x^r(4r^2 - 4r + 8r + 17) = 0$$

$$\begin{array}{r} 16 \\ \underline{17} \\ 112 \\ \underline{16} \\ 272 \end{array}$$

or $\quad 4r^2 + 4r + 17 = 0$

$$r_{1,2} = \frac{-4 \pm \sqrt{16-272}}{8} = \frac{-4 \pm i\sqrt{16}}{8}$$

or $\quad r_{1,2} = -\frac{1}{2} \pm \frac{4}{8}i = -\frac{1}{2} \pm 2i$

Therefore, the

$$\boxed{y(x) = C_1 x^{-1/2}\cos(2\ln x) + C_2\, x^{-1/2}\sin(2\ln x)}$$

If ICs: $y(1) = 2, \quad y'(1) = -3$

Then, $\quad C_1(1)^{-1/2}\cos(2\underbrace{\ln(1)}_{=0}) + C_2(1)^{-1/2}\sin(2\ln(1)) = 2$

$$C_1 + 0 = 2 \Rightarrow \boxed{C_1 = 2}$$

Also,

$$y'(x) = -\frac{1}{2} C_1 X^{-3/2} \cos(2\ln x) - C_1 \frac{2}{X} X^{-1/2} \sin(2\ln x)$$
$$- \frac{1}{2} C_2 X^{-3/2} \sin(2\ln x) + C_2 \frac{2}{X} X^{-1/2} \cos(2\ln x)$$

or

$$y'(x) = -\frac{1}{2} C_1 X^{-3/2} \cos(2\ln x) - C_1 2 X^{-3/2} \sin(2\ln x)$$
$$- \frac{1}{2} C_2 X^{-3/2} \sin(2\ln x) + C_2 2 X^{-3/2} \cos(2\ln x)$$

Using I.C. : $y'(1) = -3$

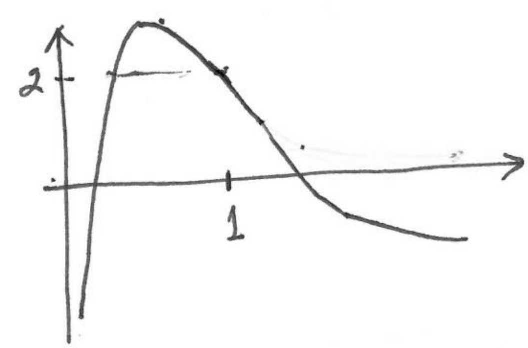
$$-\frac{1}{2} C_1 + 2C_2 = -3$$

Since $c_1 = 2$ $\Rightarrow$ $2C_2 = -3 + 1 = -2$

$$\boxed{C_2 = -1}$$

and the soln is

$$\boxed{y(x) = 2X^{-1/2} \cos(2\ln x) - X^{-1/2} \sin(2\ln x)}$$



Theorem 5.5.1 The general solution of the Euler equation (1),

$$x^2y'' + \alpha xy' + \beta y = 0,$$

in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation

$$F(r) = r(r-1) + \alpha r + \beta = 0.$$

If the roots are real and different, then

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}. \tag{24}$$

If the roots are real and equal, then

$$y = (c_1 + c_2 \ln|x|)|x|^{r_1}. \tag{25}$$

If the roots are complex conjugates, then

$$y = |x|^{\lambda}[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)], \tag{26}$$

where $r_1, r_2 = \lambda \pm i\mu$.