6.1 Definition of Laplace Transform.

Improper Integrals

0) $\int_0^\infty 1\,dt = \lim_{A\to\infty}\int_0^A 1\,dt = \lim_{A\to\infty} t\Big|_0^A = \infty$

Consider

i) $$\int_0^\infty e^{ct}dt = \lim_{A\to\infty}\frac{e^{ct}}{c}\Big|_0^A = \lim_{A\to\infty}\left[\frac{e^{cA}-1}{c}\right]$$

$$= \begin{cases} -\frac{1}{c}, & \text{if } c<0 \\ +\infty, & \text{if } c\geq 0 \end{cases}$$

ii) $$\int_1^\infty \frac{1}{t}dt = \lim_{A\to\infty} \ln(t)\Big|_1^A = \lim_{A\to\infty}\left[\ln(A) - \overset{=0}{\ln(1)}\right]$$

$$= +\infty$$

iii) $$\int_1^\infty \frac{1}{t^p}dt = \lim_{A\to\infty}\int_1^A \frac{1}{t^p}dt =$$

$$= \lim_{A\to\infty}\int_1^A t^{-p}dt = \lim_{A\to\infty}\frac{t^{-p+1}}{-p+1}\Big|_1^A$$

$$= \lim_{A\to\infty}\left[\frac{A^{1-p}}{1-p} - \frac{1}{1-p}\right] = \begin{cases} -\frac{1}{1-p} = \frac{1}{p-1}, & p>1 \\ +\infty, & p\leq 1 \quad (\text{using (ii)}) \end{cases}$$

Consider the following improper integrals:

i) $$\int_0^\infty e^{-st}\cdot 1\,dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{e^{-s(\infty)} - e^{-s(0)}}{-s}$$

$s>0$ fixed

$$= \frac{e^{-\infty} - 1}{-s} = \frac{0-1}{-5} = \frac{1}{s}, \; s>0$$

So, $\boxed{\int_0^\infty e^{-st}\cdot 1\,dt = \frac{1}{s}, \; s>0}$

ii) $$\int_0^\infty e^{-st}\cos(at)\,dt, \quad s>0$$

Before solving this integral, let us obtain

$$\int \underset{f}{e^{-t}}\,\underset{g'}{\cos t}\,dt \overset{I.P.P.}{=} \underset{f}{e^{-t}}\,\underset{g}{\sin t} - \int \underset{f'}{(-1)e^{-t}}\,\underset{g}{\sin t}\,dt \qquad (2.1)$$

Now, $$\int e^{-t}\sin t\,dt \overset{I.P.P.}{=} e^{-t}(-\cos t) - \int (-1)e^{-t}(-\cos t)\,dt$$
$$= -e^{-t}\cos t - \int e^{-t}\cos t\,dt \qquad (2.2)$$

Substituting (2.2) into (2.1)

$$\int e^{-t}\cos t\,dt = e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\cos t\,dt$$

$$\Rightarrow 2\int e^{-t}\cos t\,dt = e^{-t}(\sin t - \cos t)$$

$$\Rightarrow \boxed{\int e^{-t}\cos t\,dt = \frac{1}{2}e^{-t}(\sin t - \cos t)}$$

Now, let's follow a similar procedure to obtain

$$\text{ii)} \int_0^\infty \underset{f}{e^{-st}} \underset{g'}{\cos(at)}\,dt = e^{-st}\frac{\sin(at)}{a}\Big|_0^\infty - \int_0^\infty -s e^{-st}\frac{\sin(at)}{a}\,dt \quad , s>0$$

$$= \frac{e^{-\infty}\sin(a(\infty)) - e^{0}\sin(0)}{a} + \int_0^\infty s e^{-st}\frac{\sin(at)}{a}\,dt$$

$$= \frac{0-0}{a} + \frac{s}{a}\int_0^\infty e^{-st}\sin(at)\,dt$$

$$= \frac{s}{a}\int_0^\infty e^{-st}\sin(at)\,dt$$

$$\overset{\substack{\text{I.P.P.}\\ \text{again}}}{=} \frac{s}{a}\left[e^{-st}\frac{(-\cos(at))}{a}\Big|_0^\infty - \int_0^\infty -s e^{-st}\left(-\frac{\cos(at)}{a}\right)dt\right]$$

$$= \frac{s}{a}\left[\frac{-\overset{\to 0}{e^{-\infty}\cos(\infty)} - e^{0}\overset{=-1}{(-\cos 0)}}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)\,dt\right]$$

$$= \frac{s}{a}\left(\frac{1}{a}\right) - \frac{s^2}{a^2}\int_0^\infty e^{-st}\cos(at)\,dt$$

$$= \frac{s}{a^2} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\cos(at)\,dt$$

Therefore, if $I = \int_0^\infty e^{-st}\cos(at)\,dt$, then

$$I = \frac{s}{a^2} - \frac{s^2}{a^2} I$$

$$\Rightarrow \left(1 + \frac{s^2}{a^2}\right) I = \frac{s}{a^2}$$

$$\Rightarrow \left(\frac{a^2+s^2}{a^2}\right) I = \frac{s}{a^2}$$

or $$I = \frac{s}{a^2+s^2}, \quad s>0$$

or $$\boxed{\int_0^\infty e^{-st}\cos(at)\,dt = \frac{s}{a^2+s^2}, \quad s>0.}$$

In particular, if $a=3$.

$$\int_0^\infty e^{-st}\cos(3t)\,dt = \frac{s}{9+s^2}, \quad s>0.$$

Similarly,

iii) $$\boxed{\int_0^\infty e^{-st}\sin(at)\,dt = \frac{a}{s^2+a^2}, \quad s>0}$$

iv) $\int_0^{\infty} e^{-st} e^{at} dt, \quad s > a.$

$$\int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^{\infty} = \frac{1}{-(s-a)} \left[e^{-\infty} - e^{0} \right] = \frac{-1}{-(s-a)} = \frac{1}{s-a}$$

or $$\boxed{\int_0^{\infty} = e^{-st} e^{at} dt = \frac{1}{s-a}, \quad s > a}$$

All of these integrals are of the form

$$\boxed{\int_0^{\infty} e^{-st} f(t)\, dt = F(s)} \quad (5.1)$$

The function $f(t)$ is continuous and

$$|f(t)| \leq \begin{cases} 1, & f(t) = 1 \\ 1, & \cos(at), \sin(at) \\ e^{at}; & f(t) = e^{at} \end{cases}$$

Now, <u>we want to determine</u> what <u>conditions</u> <u>$f(t)$ needs to satisfy for (5.1) to converge</u>

Definition of Laplace Transform

<u>Def.</u>- $f(t)$ is piecewise conts on $[\alpha,\beta]$
(p.w.c)

if there exists a partition of $[\alpha,\beta)$

$$\alpha = t_0 < t_1 < \dots < t_n = \beta$$

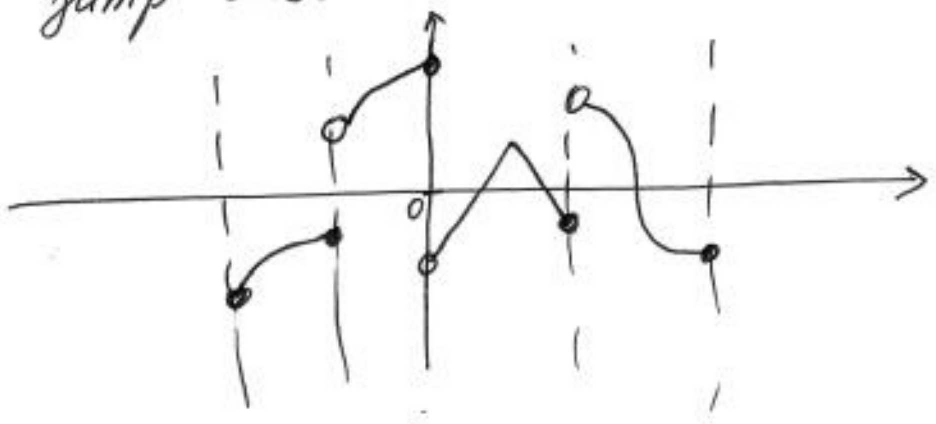
So that

a) f is continuous on each subinterval (t_{i-1}, t_i), $i = 1, .., n$.

b) f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval

<u>Equivalent to say:</u>

f is p.w.c. on $[\alpha,\beta]$ if it is continuous in this interval except for a finite number of jump discontinuities.



Thm 6.1.1 (Comparison theorem).

i) $f(t)$ p.w.c., $t \geq a$.

ii) $|f(t)| \leq g(t)$, $t \geq M > 0$

iii) $\int_M^\infty g(t)\,dt$ Converges

Then, $\int_a^\infty f(t)\,dt$ Converges

Similarly,

i) $f(t) \geq g(t) \geq 0$, $t \geq M$

ii) $\int_0^\infty g(t)\,dt$ diverges

Then, $\int_0^\infty f(t)\,dt$ diverges.

Theorem 6.1.2 If

i) $f(t)$ picewise continuous (pwc) on $0 \le t \le A$ for any $A > 0$.

ii) $|f(t)| \le K e^{at}$, $t \geq M$, $K, M > 0$ and $a \in \mathbb{R}$.

A function $f(t)$ Satisfying this Condition is Called a "function of exponential order"

Then,

$$I = \int_0^\infty e^{-st} f(t)\, dt \quad \text{exists for } s > a. \text{ (Converge)}$$

Proof.

$$\int_0^\infty e^{-st} f(t) dt = \int_0^M e^{-st} f(t) dt + \int_M^\infty e^{-st} f(t) dt$$

$$= \underbrace{\text{Constant}}_{= C_1} + \int_M^\infty e^{-st} f(t)\, dt$$

Now, from (ii)

$$\left|e^{-st} f(t)\right| \leq K\, e^{-st} e^{at} = K\, e^{(-s+a)t}, \quad t \geq M$$

Then,

$$\int^{\infty} K e^{(a-s)t}\, dt = \lim_{A\to\infty} \int_M^A K e^{(a-s)t} dt$$

$$= K \lim_{A\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \Big|_M^A \right]$$

$$= \frac{K}{a-s} \lim_{A\to\infty} \left[e^{(a-s)A} - e^{(a-s)M} \right]$$

If $s > a$

$$= K\, \frac{e^{(a-s)M}}{s-a} = C_2, \quad s > a$$

Therefore, $\displaystyle\int_0^{\infty} K e^{(a-s)t} = C_1 + C_2$ Converges

Comparison theorem implies

$\displaystyle\int_0^{\infty} e^{-st} f(t)\, dt$ also converges.

By comparison theorem

$$\int_0^{\infty} e^{-st} f(t)\, dt \quad \text{converges.}$$

Definition (Laplace transform)

If $f(t)$ satisfies conditions (i) and (ii) of thm 6.1.2:

i) $f(t)$ is p.w.c. on $0 \le t \le A$, for any $A > 0$.

ii) $f(t)$ is of exponential order.

Then,

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt, \quad s > a$$

is called the Laplace transform of $f(t)$.

The Laplace transform can be seen as an integral operator:

$$\mathcal{L} : \underset{\text{of exp order}}{\{\text{p.w.c.}\}} \longrightarrow \text{set of functions}$$

$$f(t) \longrightarrow \mathcal{L}\{f(t)\} = F(s).$$

Linearity of $\mathcal{L}$

Assume $f_1(t)$ and $f_2(t)$ have Laplace transforms for $s > a_1$ and $s > a_2$, respectively

then for $s > \max(a_1, a_2)$

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}$$

$$= \int_0^\infty (c_1 f_1(t) + c_2 f_2(t)) e^{-st} dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t) dt + c_2 \int_0^\infty e^{-st} f_2(t) dt$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.$$

Therefore, the Laplace transform is a linear operator.

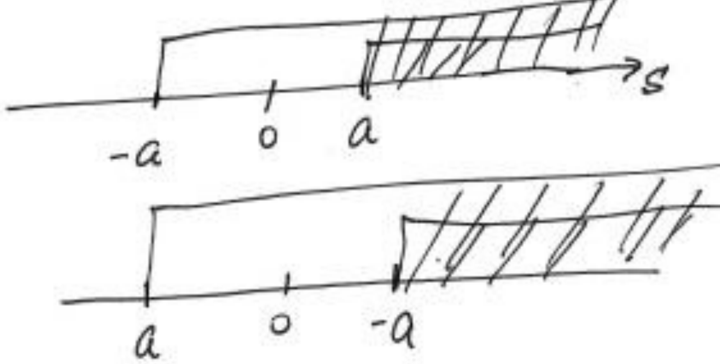
Application:

v) $\mathcal{L}\{\cosh(at)\} = \mathcal{L}\left\{\dfrac{e^{at}+e^{-at}}{2}\right\}$

linearity

$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$

$= \dfrac{1}{2}\,\dfrac{1}{s-a} + \dfrac{1}{2}\,\dfrac{1}{s+a}, \quad \begin{matrix} s>a \\ s>-a \end{matrix}, \text{ if } a>0 \Downarrow s>a$



If $a<0$, $\begin{matrix} s>a \\ s>-a \end{matrix} \Rightarrow s>-a$

Both cases $s>|a|$.

$= \dfrac{\frac{1}{2}(s+a)+\frac{1}{2}(s-a)}{(s-a)(s+a)} = \dfrac{s}{s^2-a^2};$

Therefore,

$$\boxed{\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}, \quad s>|a|}$$

Similarly,

VI) $$\mathcal{L}\{\sinh(at)\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left(\frac{s+a-s+a}{s^2-a^2}\right)$$

$$= \frac{a}{s^2-a^2}, \quad s>|a|$$

Then,

$$\boxed{\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}, \quad s>|a|.}$$

Exercise 22 (Sect. 6.1).

$\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 \underbrace{e^{-st}}_{g'}\underbrace{t}_{f}\,dt = t\frac{e^{-st}}{-s}\Big|_0^1$$

$$-\int_0^1 (1)\frac{e^{-st}}{-s}\,dt = \frac{e^{-s}}{-s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\Big|_0^1\right]$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s}\left[\frac{e^{-s}}{-s} - \frac{1}{-s}\right] = \frac{-e^{-s}}{s} - \frac{1}{s^2}\left(e^{-s}-1\right)$$

$$= \boxed{\frac{1-(s+1)e^{-s}}{s^2}}$$