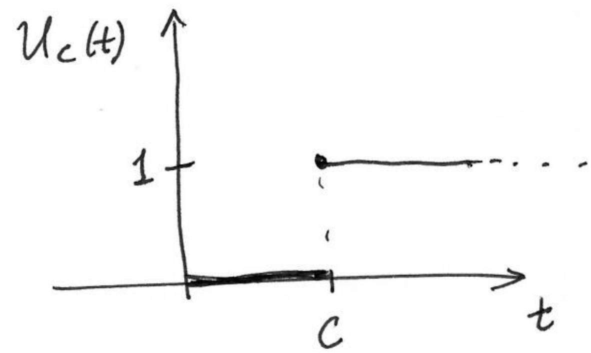
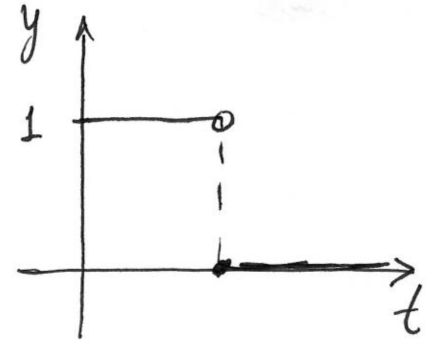
6.3 Step functions

Unit step or Heaviside function

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

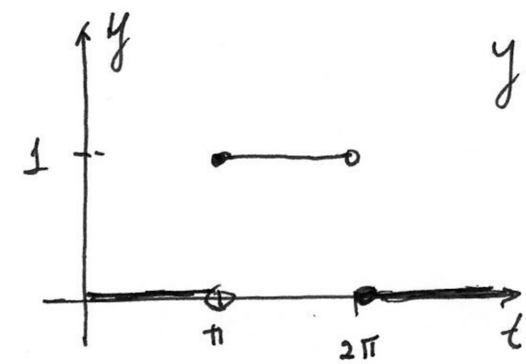


Using this function, we can define many other functions with jump discontinuity.



$y(t) = 1 - u_c(t)$

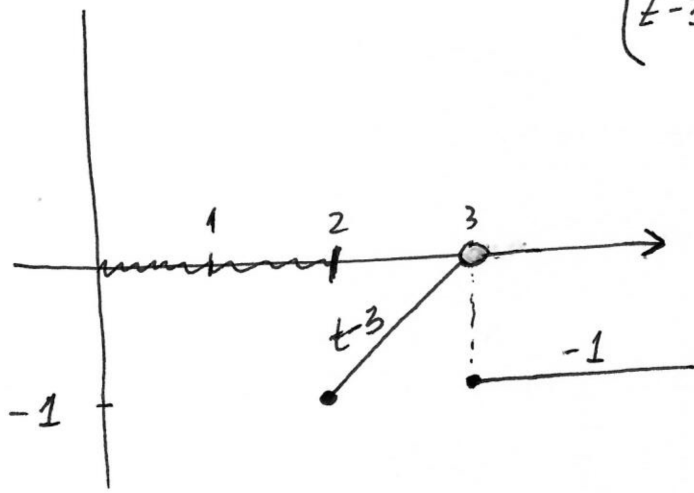
or



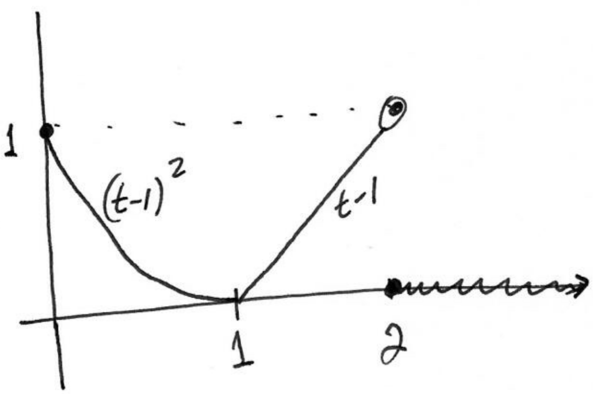
$y(t) = u_\pi(t) - u_{2\pi}(t)$

IV Graph: $(t-3)U_2(t) - (t-2)U_3(t)$

$$(t-3)U_2(t) - (t-2)U_3(t) = \begin{cases} 0, & 0 \le t < 2 \\ t-3 & 2 \le t < 3 \\ t-3-t+2 = -1, & t \ge 3 \end{cases}$$



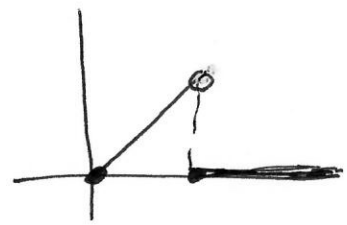
IV The graph below corresponds to which function.



$$g(t) = (t-1)^2 - U_1(t)(t-1)^2 + \\ + (t-1)U_1(t) - (t-1)U_2(t)$$

In homework problem 25 Sect 6.2,

$$y'' + y = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases} = f(t)$$



$$\boxed{f(t) = t - t\,u_1(t)}$$

Laplace transform of $u_c(t)$, for $c \geq 0$.

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt =$$

$$= \int_c^\infty e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_c^\infty = \frac{e^{-cs}}{s}, \quad s > 0.$$

Thm 6.3.1 1) $F(s) = \mathcal{L}\{f(t)\}$, $s > a \geq 0$,

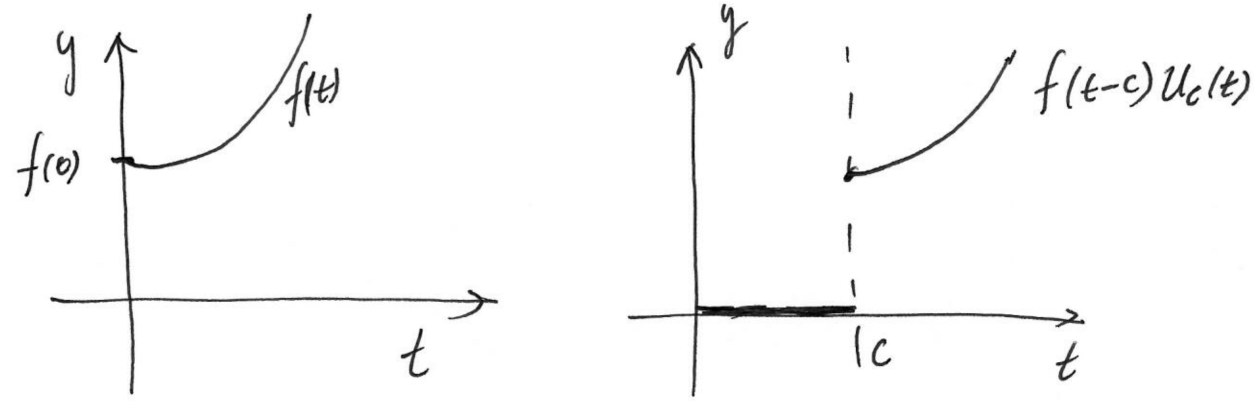
2) $c > 0$.

Then $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

$$= e^{-cs}F(s), \quad s > a$$

Proof $\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$

Before Consider



$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

$\xi = t - c$

$t = \xi + c$

$dt = d\xi$

$$\int_0^\infty e^{-s(\xi+c)} f(\xi)\, d\xi$$

$$= e^{-cs} \int_0^\infty e^{-s\xi} f(\xi)\, d\xi = e^{-cs} F(s).$$

Notice $\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t)\, f(t-c)$

Laplace Transform of

$$g(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$$

$$g(t) = u_2(t)\,(t-2)^2$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_2(t)\,(t-2)^2\}$$

Using previous thm. 6.3.1

$c = 2, \quad f(t) = t^2 \Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} = \dfrac{2}{s^3}, \; s > 0$

Then, $\mathcal{L}\{u_2(t)\,(t-2)^2\} = e^{-2s}\dfrac{2}{s^3}, \quad s > 0.$

Application of previous Thm.:

Find the Lapl. Transf. of

$$f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 1, & t \geq 1 \end{cases}$$

$$t^2 - 2t + 1 = (t-1)^2 \Rightarrow f(t) = u_1(t)\,(t-1)^2$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)\,(t-1)^2\} \overset{\text{thm}}{=} e^{-s}\mathcal{L}\{t^2\} = e^{-s}\frac{2}{s^3}$$

#8 (6.3)

$$g(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$$

$$t^2 - 2t + 2 = t^2 - 2t + 1 + 1 = (t-1)^2 + 1$$

$$\Rightarrow g(t) = u_1(t)\left[(t-1)^2 + 1\right] = u_1(t)\,(t-1)^2 + u_1(t)$$

$$\Rightarrow \mathcal{L}\{g(t)\} = \mathcal{L}\{u_1(t)\,(t-1)^2\} + \mathcal{L}\{u_1(t)\}$$

$$= e^{-s}\frac{2}{s^3} + \frac{e^{-s}}{s} = e^{-s}\left[\frac{1}{s} + \frac{2}{s^3}\right]$$

Thm 6.3.2

1) $F(s) = \mathcal{L}\{f(t)\}, \quad s > a \geq 0,$

2) c is a constant

then,

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \quad s > a+c$$

Proof.

$$\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-c)t} f(t)\,dt = F(s-c), \quad s > a+c$$

IV So $\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\} = e^{2t}t^3$, since $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$

Notice the difference between

$$\mathcal{L}\{e^{2t}t^3\} = \frac{3!}{(s-2)^4}, \quad s>2$$

and

$$\mathcal{L}\{u_2(t)\,(t-2)^3\} = e^{-2s}\frac{3!}{s^4}, \quad s>0.$$

III Important application:

a) $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}, \quad s>a.$

b) $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}, \quad s>a.$

c) $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}, \quad s>a.$

Ex. #17

omit if no time

$$G(s) = \frac{(s-2)\,e^{-s}}{s^2-4s+3} = e^{-s}\frac{s-2}{s^2-4s+3}$$

$$= e^{-s}\frac{s-2}{(s-3)(s-1)} = e^{-s}\left[\frac{A}{s-3}+\frac{B}{s-1}\right] =$$

$A = 1/2$
$B = 1/2$

$$= \frac{1}{2}e^{-s}\left[\frac{1}{s-3}+\frac{1}{s-1}\right] \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s-3}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s-1}\right\}$$

$$= \frac{1}{2}u_1(t)\,e^{3(t-1)} + \frac{1}{2}u_1(t)\,e^{1(t-1)}.$$

Turn

For training on Laplace Transform:

i) $\mathcal{L}\{\sinh(at)\} = \dfrac{a}{s^2-a^2}, \quad s > |a|$

ii) $\mathcal{L}\{e^{ct}f(t)\} = F(s-c).$

Application of (i)-(ii).

6.2
#20 Find the inverse Laplace transform of

$$F(s) = \frac{1}{9s^2-12s+3}$$

Answ: $\dfrac{1}{9s^2-12s+3} = \dfrac{1}{9\left(s^2-\frac{4}{3}s+\frac{1}{3}\right)} = \dfrac{1/9}{\left(s-\frac{2}{3}\right)^2-\frac{1}{9}}$

$$= \frac{1/9}{\left(s-2/3\right)^2-\left(1/3\right)^2} = \frac{1}{3}\,\frac{1/3}{\left[(s-2/3)^2-\left(\frac{1}{3}\right)^2\right]}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{e^{2/3\,t}}{3}\sinh\left(t/3\right).$$

Direct: $\mathcal{L}\{\sinh t\} = \dfrac{1}{s^2-1}, \quad \mathcal{L}\{\sinh bt\} = \dfrac{1}{s^2-b^2}$

$\mathcal{L}\{e^{at}\sinh bt\} = \dfrac{1}{(s-a)^2-b^2}$

How about ?

$$\mathcal{L}^{-1}\left\{e^{-6s}\frac{1}{9s^2-12s+3}\right\}$$

answ: $$\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{3}\frac{1/3}{(s-2/3)^2-(1/3)^2}\right\}$$

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}$$

$$\mathcal{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathcal{L}\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$$

$$\mathcal{L}\{u_c(t)\,e^{a(t-c)}\sinh b(t-c)\} = e^{-cs}\frac{b}{(s-a)^2-b^2}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{3}\frac{1/3}{(s-2/3)^2-(1/3)^2}\right\} = \frac{1}{3}u_6(t)e^{2/3(t-6)}\sinh\left(\frac{t-6}{3}\right)$$

Summarizing.

Thm. 6.3.1 $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s)$. and $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t) f(t-c)$

Thm 6.3.2. $\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$. and $\mathcal{L}^{-1}\{F(s-c)\} = e^{ct} f(t)$

Some examples

(I) $\dfrac{As+B}{s^2+bs+c}$,

Examples:

$$\frac{5s+3}{s^2-6s+34} = \frac{5s+3}{(s-3)^2+25} = \frac{5(s-3)+15+3}{(s-3)^2+25} =$$

$$= 5\,\frac{s-3}{(s-3)^2+25} + \left(\frac{18 * 5}{(s-3)^2+25}\right)\frac{1}{5}$$

$$\xrightarrow{\mathcal{L}^{-1}} 5e^{3t}\cos(5t) + \frac{18}{5}e^{3t}\sin(5t).$$

(II) $\dfrac{e^{-cs}(As+B)}{s^2+bs+C}$,

Examples:

$$\frac{e^{-5s}(2s+6)}{s^2-2s+10} = e^{-5s}\left[\frac{2s+6}{(s-1)^2+9}\right] =$$

$$= e^{-5s}\,\frac{2(s-1)+6+2}{(s-1)^2+9} = 2e^{-5s}\,\frac{s-1}{(s-1)^2+9} + \frac{8}{3}\,\frac{3e^{-5s}}{(s-1)^2+9}$$

$$\xrightarrow{\mathcal{L}^{-1}} 2u_5(t)\,e^{t-5}\cos(3(t-5)) + \frac{8}{3}u_5(t)\,e^{t-5}\sin(3(t-5))$$