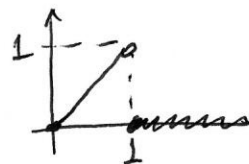


6.4 ODE's with discontinuous forcing functions.

①

Back to problem 25 (6.2)



$$y'' + y = \begin{cases} t, & 0 \leq t \leq 1 \\ 0, & 1 \leq t < \infty \end{cases} \equiv f(t), \quad \begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

$\mathcal{L}\{f(t)\}$ was obtained in section 6.2 using the definition directly.

Now, we can represent $f(t)$ using step functions and then use thm. of 6.3. to obtain $\mathcal{L}\{f(t)\}$.

In fact,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t - t u_1(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)t\} = \\ &= \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)[(t-1)+1]\} = \\ &= \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)(t-1)\} - \mathcal{L}\{u_1(t)\} = \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} = \frac{1 - e^{-s} - s e^{-s}}{s^2} = \\ &= \frac{1}{s^2} - \frac{e^{-s}(s+1)}{s^2} \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{L}\{y''+y\} &= s^2 Y(s) - s y'(0) - y'(0) + Y(s) \\ &= (s^2+1) Y(s) \end{aligned}$$

$$\therefore \boxed{Y(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}(s+1)}{s^2(s^2+1)}}$$

Partial fraction decomposition:

$$\begin{aligned} \frac{1}{s^2(s^2+1)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{A(s^2+1)s + B(s^2+1) + s^2(Cs+D)}{s^2(s^2+1)} \\ &= \frac{As^3 + As + Bs^2 + B + Cs^3 + Ds^2}{s^2(s^2+1)} \end{aligned}$$

$$\begin{array}{lcl} s^3: & A + C & = 0 \quad \boxed{C=0} \\ s^2: & B + D & = 0 \quad \boxed{D=-1} \\ s^1: & A & = 0 \quad \boxed{A=0} \\ s^0: & B & = 1 \Rightarrow \boxed{B=1} \end{array}$$

$$\therefore \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Similarly,

$$\frac{S+1}{S^2(S^2+1)} = \frac{A}{S} + \frac{B}{S^2} + \frac{CS+D}{S^2+1}$$

$$S^3: A + C = 0 \Rightarrow \boxed{C = -1}$$

$$S^2: B + D = 0 \Rightarrow \boxed{D = -1}$$

$$S: \boxed{A = 1}$$

$$S^0: \boxed{B = 1}$$

$$\therefore \frac{S+1}{S^2(S^2+1)} = \frac{1}{S} + \frac{1}{S^2} - \frac{S+1}{S^2+1}$$

$$\therefore Y(s) = \frac{1}{S^2(S^2+1)} - \frac{e^{-s}(S+1)}{S^2(S^2+1)}$$

$$\text{or } Y(s) = \frac{1}{S^2} - \frac{1}{S^2+1} - \frac{e^{-s}}{S} - \frac{e^{-s}}{S^2} + \underbrace{\frac{e^{-s}(S+1)}{S^2+1}}_{e^{-s} \frac{S}{S^2+1} + e^{-s} \frac{1}{S^2+1}}$$

$$\mathcal{L}^{-1} \downarrow$$

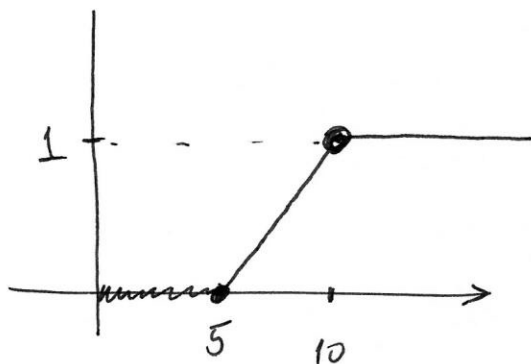
$$y(t) = t - \sin t - \cancel{u_1(t)} - \cancel{u_1(t)(t-1)} + u_1(t) \cos(t-1) + u_1(t) \sin(t-1)$$

$$\text{or } \boxed{y(t) = t - \sin t - t u_1(t) + u_1(t) [\cos(t-1) + \sin(t-1)]}$$

Example 2.- (book).

$$y'' + 4y = g(t) = \begin{cases} 0, & 0 \leq t < 5 \\ \frac{t-5}{5}, & 5 \leq t < 10 \\ 1, & t \geq 10 \end{cases} \quad \begin{cases} y(10) = 0 \\ y'(10) = 0 \end{cases}$$

First, represent $g(t)$ using step functions:



$$g(t) = u_5(t) \frac{t-5}{5} - u_{10}(t) \frac{t-5}{5} + u_{10}(t)$$

$$= u_5(t) \frac{t-5}{5} - u_{10}(t) \left[\frac{t-5}{5} - 1 \right] =$$

$$= u_5(t) \frac{t-5}{5} - u_{10}(t) \left(\frac{t-10}{5} \right)$$

or

$$g(t) = \frac{1}{5} \left[u_5(t) (t-5) - u_{10}(t) (t-10) \right]$$

$$\Rightarrow \mathcal{L}\{g(t)\} = \frac{1}{5} \mathcal{L}\{u_5(t)(t-5)\} - \frac{1}{5} \mathcal{L}\{u_{10}(t)(t-10)\}$$

$$\Rightarrow G(s) = \frac{1}{5} e^{-5s} \frac{1}{s^2} - \frac{1}{5} e^{-10s} \frac{1}{s^2}$$

Applying Lapl. Transf. to the IVP:

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$$

$$\text{or } s^2 Y(s) - s y(0) - y'(0) + 4 Y(s) = G(s)$$

$$\Rightarrow Y(s) = \frac{1}{5} \left\{ \frac{e^{-5s}}{s^2(s^2+4)} - \frac{e^{-10s}}{s^2(s^2+4)} \right\}$$

$$\text{If } H(s) \equiv \frac{1}{s^2(s^2+4)}$$

$$\text{then } Y(s) = \frac{1}{5} \left\{ e^{-5s} H(s) - e^{-10s} H(s) \right\}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{5} \left[u_5(t) h(t-5) - u_{10}(t) h(t-10) \right]} \quad (*)$$

$$\text{where } h(t) = \mathcal{L}^{-1}\{H(s)\}.$$

Let's find $h(t)$

$$H(s) = \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

Solving linear syst.
 $A=0, B=1/4, C=0, D=-1/4.$

$\frac{1}{8} \frac{2}{s^2+4}$

$$\Rightarrow \downarrow \mathcal{L}^{-1} \quad h(t) = \frac{1}{4}t - \frac{1}{8} \sin(2t)$$

$$\Rightarrow h(t-5) = \frac{1}{4}(t-5) - \frac{1}{8} \sin(2(t-5))$$

$$h(t-10) = \frac{1}{4}(t-10) - \frac{1}{8} \sin(2(t-10))$$

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Subst. into (*)

$$f(t) = \frac{1}{20} \left\{ u_5(t) (t-5) - u_{10}(t) (t-10) \right\} \\ - \frac{1}{40} \left\{ u_5(t) \sin(2(t-5)) - u_{10}(t) \sin(2(t-10)) \right\}$$