6.6 Convolution Integral

We are interested in a formula for

$$\mathcal{L}^{-1}\{F(s)\,G(s)\}$$

Our first reaction may be to claim that

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f(t)\,g(t)$$

Where $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$

For example, if

$$F(s) = G(s) = \frac{1}{s-a}$$

then, we may be tempted to say

that
$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\cdot\frac{1}{s-a}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$= e^{at}\cdot e^{at} = e^{2at}. \quad ?$$

But, we know from thm. 6.3.2

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c). \quad , \quad s > a+c \quad (2.1)$$

Then,

$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t). \qquad (2.2)$$

we also know, $\mathcal{L}\{t\} = \frac{1}{s^2}$ Table 6.2.1 #3.

Therefore, applying (2.2)

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = e^{at}t.$$

Thm 6.6.1

1) $F(s) = \mathcal{L}\{f(t)\}$
$G(s) = \mathcal{L}\{g(t)\}$, exist for $s > a \geq 0$

Then,
$$H(s) = F(s)\,G(s) = \mathcal{L}\{h(t)\}, \quad s > a$$

where
$$h(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

h is known as the convolution of f and g.

the integrals are called convolution integrals.

Short notation:
$$h(t) = f(t) * g(t).$$

Using thm 6.6.1

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = h(t)$

Notice that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} = f(t)$

Then $h(t) = \int_0^t f(t-\tau)\, f(\tau)\, d\tau$

$$= \int_0^t e^{a(t-\tau)} e^{a\tau}\, d\tau$$

$$= \int_0^t e^{at-a\tau} \cdot e^{a\tau}\, d\tau$$

$$= \int_0^t e^{at} e^{(a\tau-a\tau)}\, d\tau = \int_0^t e^{at}\, d\tau$$

$$= e^{at}\tau\Big|_0^t = t e^{at}.$$

or $\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = t e^{at}}$ ✓ Check with Table 6.2.1 # 11.

Application of Convolution theorem to IVP with constant coefficients.

Consider
$$\begin{cases} ay'' + by' + cy = g(t) & (1.1) \\ y(0) = y_0, \quad y'(0) = y_0' & (1.2) \end{cases}$$

This problem can be separated into two IVPs.

In fact,
$$y(t) = y_1(t) + y_2(t)$$

where y_1 satisfies
$$\begin{cases} a y_1'' + by_1' + cy_1 = 0 & (1.3) \\ y_1(0) = y_0, \quad y_1'(0) = y_0' & (1.4) \end{cases}$$

and y_2 satisfies
$$\begin{cases} a y_2'' + by_2' + cy_2 = g(t) & (1.5) \\ y(0) = 0, \quad y'(0) = 0 & (1.6) \end{cases}$$

The IVP (1.1) - (1.2) is referred as

input - Output problem

where

① Coefficients a, b, c properties of Physical Sistem.

② $g(t)$ is the input to the system

③ y_0, y_0' describe the initial state

④ the Soln. $y(t)$ is the output at time t.

If we apply L.T. to these two problems, then we obtain

$$a\left(s^2 Y_1(s) - s y_0 - y_0'\right) + b\left(s Y_1(s) - y_0\right) + c Y_1(s) = 0$$

or

$$(as^2 + bs + c) Y_1(s) = (as + b) y_0 + a y_0'$$

Solving for $Y_1(s)$

$$\boxed{Y_1(s) = \frac{(as+b) y_0 + a y_0'}{as^2 + bs + c}}$$

Also,

$$\mathcal{L}\{a y_2'' + b y_2' + c y_2\} = \mathcal{L}\{g(t)\}$$

or

$$(as^2 + bs + c) Y_2(s) = G(s)$$

Then,

$$\boxed{Y_2(s) = \frac{G(s)}{as^2 + bs + c}}$$

On the other hand, if we apply L.T. to (1.1) and (1.2)

$$\mathcal{L}\{(1.1), (1.2)\}$$

$$a\left(s^2 Y(s) - s y_0 - y_0'\right) + b\left(sY(s) - y_0\right) + cY(s) = G(s)$$

Then, $(as^2 + bs + c)Y(s) = (as+b)y_0 + ay_0' + G(s)$

and

$$\boxed{\begin{aligned} Y(s) &= \underbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}_{=\,Y_1(s)} + \underbrace{\frac{G(s)}{as^2+bs+c}}_{=\,Y_2(s)} \\ \text{or}\quad Y(s) &= Y_1(s) + Y_2(s) \end{aligned}}$$

Then, to obtain $y(t)$, we can obtain $y_1(t)$ taking

(I) $$\boxed{\mathcal{L}^{-1}\{Y_1(s)\} = y_1(t)}$$

To do this, we use P.F.D. and Table 6.2.1.

and

(II) $\mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\{H(s)\,G(s)\}$

where $H(s) = \dfrac{1}{as^2+bs+c}$.

is called transfer function. (properties of system)

Also, $G(s)$ depends on the external force $g(t)$.

Using convolution theorem

$$y_2(t) = \mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\{H(s)\,G(s)\}$$

$$= \int_0^t h(t-\tau)\,g(\tau)\,d\tau$$

A physical interpretation of $h(t)$ can be obtained by considering

$$\begin{cases} a\hat{y}'' + b\hat{y}' + c\hat{y} = \delta(t) & (5.1) \\ \hat{y}(0) = 0, \quad \hat{y}'(0) = 0 & (5.2) \end{cases}$$

Response of the system to a unit impulse applied at $t=0$.

$\hat{y}(t)$: Impulse response of the system.

Applying $\mathcal{L}\{(5.1)-(5.2)\}$

$$(as^2 + bs + c)Y(s) = \mathcal{L}\{\delta(t)\} = \int_0^\infty e^{-st}\delta(t)dt = 1$$

$$\Rightarrow \quad Y(s) = \frac{1}{as^2+bs+c} = H(s)$$

$\Rightarrow h(t) = \hat{y}(t)$: Impulse response

Then,

$$y_2(t) = \int_0^t h(t-\tau)\, g(\tau)\, d\tau$$

is the convolution of the ① impulse response $h(t)$ and ② the forcing function.

6.6 #16) $\begin{cases} y''' + y' + \frac{5}{4} y = 1 - u_\pi(t) \\ y(0) = 1, \quad y'(0) = -1 \end{cases}$

L.T.

$$\left(s^2 + s + \frac{5}{4}\right) Y(s) - s + 1 = \frac{1}{s} - \frac{e^{-\pi s}}{s} = G(s)$$

$$Y(s) = \frac{s-1}{s^2+s+5/4} + \frac{G(s)}{s^2+s+5/4} = Y_1(s) + Y_2(s)$$

Now, $Y_1(s) = \dfrac{s-1}{s^2+s+5/4} = \dfrac{s-1}{(s+1/2)^2 - \frac{1}{4} + \frac{5}{4}} = \dfrac{s-1}{(s+1/2)^2 +}$

$$= \frac{s}{(s+1/2)^2+1} - \frac{1}{(s+1/2)^2+1}$$

$$\Rightarrow \boxed{y_1(t) = e^{-t/2}\cos(t) - e^{-t/2}\sin(t)}$$

$$Y_2(s) = G(s)H(s) = \left(\frac{1}{s} - \frac{e^{-\pi s}}{s}\right)\left(\frac{1}{s^2+s+5/4}\right)$$

$$g(t) = 1 - u_\pi(t)$$

$$\Rightarrow \boxed{y_2(t) = \int_0^t \left(1 - u_\pi(\tau)\right)\left(e^{(t-\tau)/2} \sin(t-\tau)\right) d\tau}$$

$$\boxed{y(t) = y_1(t) + y_2(t)}$$

III

input	Impulse response	Output
$f(t)$	$\hat{y}(t)$	$y(t)$

$$y(t) = \hat{y}(t) * f(t)$$

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{\ \} = \hat{Y}(s)\,F(s)$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\{\hat{Y}(s)\,F(s)\} = \int_0^t \hat{y}(t-\tau)\, f(\tau)\, d\tau .$$

Convenience ~~from~~ when input is changed (external force) but rhs of equation remains.

input	System function	output
$F(s)$	$\hat{Y}(s)$	$Y(s) = F(s)\,\hat{Y}(s).$

What is the System function?

Calculation?