7.5 Homogeneous Linear Systems with constant coefficients.

We want to study how to obtain solns. for

$$\vec{X}'(t) = A\vec{x}(t) \qquad (1.1)$$

We will assume $\det(A) \neq 0$.

Consider the scalar equ.

$$x'(t) = a\,x(t)$$

with general soln:

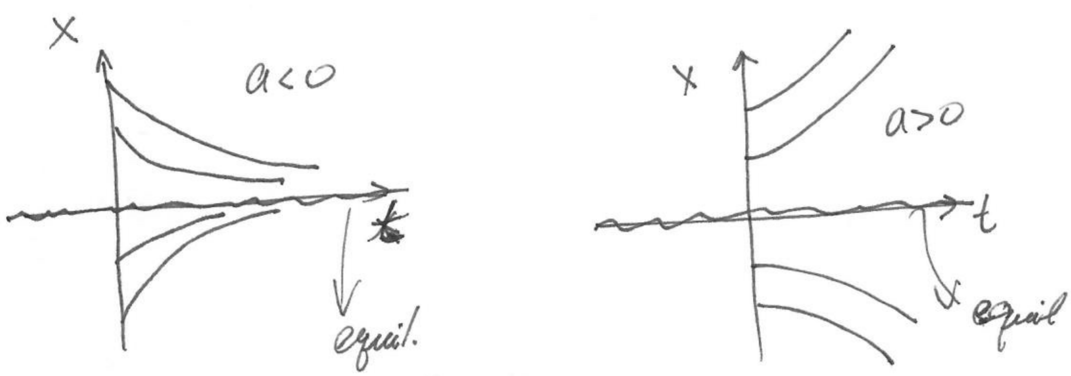
$$x(t) = C e^{at}, \quad t \in \mathbb{R}.$$

$x(t) \equiv 0$ is a soln. This is also an equilibrium soln. (Constant for any t).

In previous Chapters, we discussed Stability

a) If $a > 0$, $x(t) \equiv 0$ is an unstable equil.

b) If $a < 0$, $x(t) \equiv 0$ is a stable equil.



Equil. Soln. for (1.1) when $\frac{d\vec{x}}{dt}(t) = 0$, for any t.

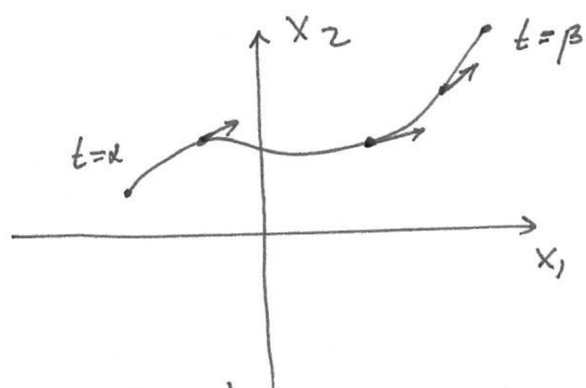
$$A\vec{x}(t) = \vec{0},$$

Since $\det A \neq 0 \Rightarrow$ only Soln. is
$$\vec{x}(t) \equiv \vec{0}.$$

Question: Is this equil. stable, or unstable?

<u>Direction Field for $A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, when $A_{2\times 2}$ matrix.</u>

The vector function
$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad t \in (\alpha, \beta)$$
represents a curve in the plane x_1x_2-plane.



$\vec{x}'(t_0) = \begin{pmatrix} x_1'(t_0) \\ x_2'(t_0) \end{pmatrix}$ is the tangent vector at $t = t_0$.

Since $\vec{x}'(t) = A\vec{x}$ for a 1st order linear system, if A 2x2 matrix we can evaluate $A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in a large number of points $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of plane $x_1 x_2$. to obtain a direction field of tangent vectors to the solns. $\vec{x}(t)$ of the Linear Syst. $\vec{x}'(t) = A\vec{x}(t)$.

SHOW MAPLE for

Example 1. $\vec{x}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

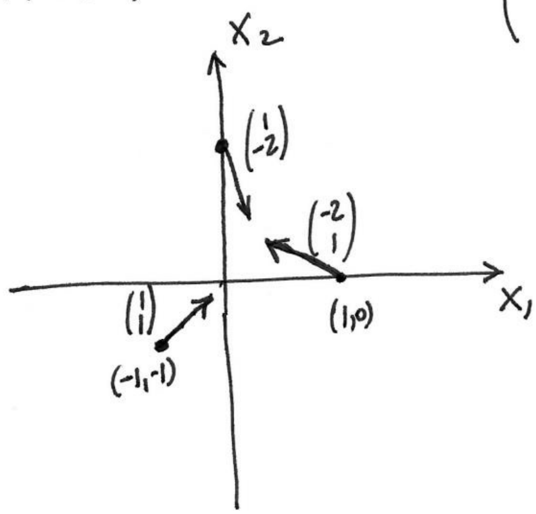
By hand,

at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



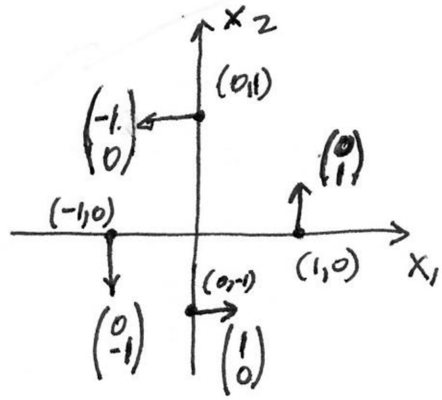
<u>Example 2</u> Consider $\quad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

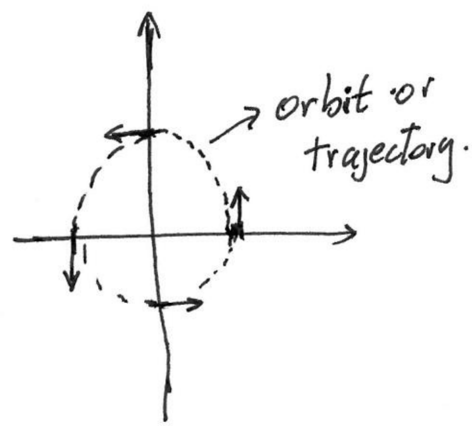
$$\vec{x}'(t) = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (3.1)$$

or $\vec{x}'(t) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

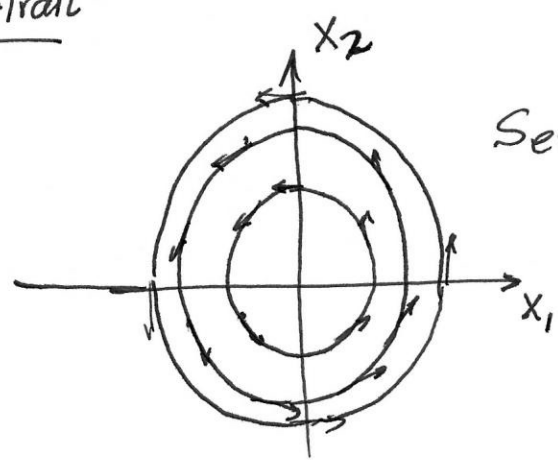
or $\quad x_1'(t) = -x_2, \quad x_2'(t) = x_1$

<u>Direction Field</u> in phase plane $\quad x_1x_2$-plane





<u>Phase Portrait</u>



Several orbits.

A vector function $\vec{X}^{(1)}(t)$ Satisfying (3.1) is given by

$$\boxed{\vec{X}^{(1)}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} X_1^{(1)} \\ X_2^{(1)} \end{pmatrix}}$$

Since $\left(\vec{X}^{(1)}(t)\right)' = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} -X_2^{(1)} \\ X_1^{(1)} \end{pmatrix}$ ✓

Another one is given by

$$\boxed{\vec{X}^{(2)}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} = \begin{pmatrix} X_1^{(2)} \\ X_2^{(2)} \end{pmatrix}}$$

Since $\left(\vec{X}^{(2)}(t)\right)' = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} -X_2^{(2)} \\ X_1^{(2)} \end{pmatrix}$

They are linearly independent in $\mathbb{R}$.

In fact,

$$W[\vec{X}^{(1)}, \vec{X}^{(2)}](t) = \begin{vmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{vmatrix} = -\cos^2 t - \sin^2 t$$
$$= -1 \neq 0$$

Therefore, using theorem 7.4.2, the general Solution Can be expressed as

$$\vec{X}(t) = C_1 \vec{X}^{(1)}(t) + C_2 \vec{X}^{(2)}(t)$$

$$= C_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

$$= \begin{pmatrix} C_1 \cos t + C_2 \sin t \\ C_1 \sin t - C_2 \cos t \end{pmatrix} = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$$

Now, we can easily prove that <u>the orbits are Circles</u>. In fact,

$$X_1^2 + X_2^2 = (C_1 \cos t + C_2 \sin t)^2 + (C_1 \sin t - C_2 \cos t)^2$$

$$= C_1^2 \cos^2 t + C_2^2 \sin^2 t + \cancel{2C_1C_2 \cos t \sin t}$$

$$+ C_1^2 \sin^2 t + C_2^2 \cos^2 t - \cancel{2C_1C_2 \cos t \sin t}$$

$$= C_1^2(\cos^2 t + \sin^2 t) + C_2^2(\sin^2 t + \cos^2 t)$$

$$= C_1^2 + C_2^2.$$

So $$\boxed{X_1^2 + X_2^2 = C_1^2 + C_2^2}$$

Construction of Solutions

Consider $$\vec{x}'(t) = A\,\vec{x}(t) \qquad (6.1)$$

Inspired in the scalar case for 2^{nd} order linear equations with constant coefficients, we look for solutions like

$$\vec{x}(t) = \vec{\xi}\, e^{rt}, \qquad \vec{\xi} = ?,\ r = ? \qquad (6.2)$$

Taking derivative

$$\vec{x}'(t) = \left[e^{rt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right]' = r e^{rt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = r e^{rt} \vec{\xi}$$

Also, $$A\vec{x}(t) = A\left(e^{rt}\vec{\xi}\right)$$

Then, for $\vec{x}'(t) = A\,\vec{x}(t)$

It should be that

$$\vec{x}'(t) = A\,\vec{x}(t)$$

$$r \cancel{e^{rt}} \vec{\xi} = \cancel{e^{rt}} A \vec{\xi}$$

$$\Rightarrow \quad \boxed{A\vec{\xi} = r\vec{\xi}} \qquad \vec{\xi} \text{ and } r \text{ are unknown}$$

(6.3)

So, we have arrived to an eigenvalue problem for the matrix A.

As a consequence, a solution of

$$\boxed{\vec{x}\,'(t) = A\vec{x}(t)}$$

Can be obtained as

$$\boxed{\vec{x}(t) = \vec{\xi}\, e^{rt}}$$

if <u>r is an eigenvalue of A</u> with <u>$\vec{\xi}$ as a corresponding eigenvector</u>.

<u>Back to example 1</u>

$$\vec{x}\,'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (7.1)$$

<u>Eigenvalue Problem</u> $\quad A\vec{x} = \lambda\vec{x}$ or $(A - \lambda I)\vec{x} = 0$

Nontrivial Solution if

$$\det(A - \lambda I) = 0$$

or

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \begin{aligned} &(-2-\lambda)(-2-\lambda) - 1 = 0 \\ &(\lambda+2)^2 - 1 = 0 \\ &\lambda^2 + 4\lambda + 4 - 1 = 0 \\ &\lambda^2 + 4\lambda + 3 = 0 \\ &(\lambda+3)(\lambda+1) = 0 \end{aligned}$$

So $\lambda_1 = -3, \qquad \lambda_2 = -1$

Let us find the corresponding eigenvectors.

For $\underline{\underline{\lambda_1 = -3}}$ $\qquad A\vec{\xi}^{(1)} = -3\vec{\xi}^{(1)} \Leftrightarrow (A - 3I)\vec{\xi}^{(1)} = \vec{0}$

$$\begin{bmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \xi_1^{(1)} = -\xi_2^{(1)}$$

$$\vec{\xi}^{(1)} = c\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad c \in \mathbb{R}.$$

Therefore a solution of (6.1) is given by

$$\boxed{\vec{x}^{(1)}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} = \begin{pmatrix} -e^{-3t} \\ e^{-3t} \end{pmatrix}}$$

For $\underline{\underline{\lambda_2 = -1}}$ $\qquad (A - (-1)I)\vec{\xi}^{(2)} = \vec{0} \Leftrightarrow (A + I)\vec{\xi} = \vec{0}$

$$\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} \vec{\xi}^{(2)} = \begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} \begin{bmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow -\xi_1^{(2)} + \xi_2^{(2)} = 0$$

$$\xi_2^{(2)} = \xi_1^{(2)}$$

$$\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

Therefore, a second solution is given by

$$\vec{X}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

This is the procedure to follow in general to obtain the solutions of any 1st order linear system.

We have shown before that

$$\vec{X}^{(1)}(t) = \begin{pmatrix} -e^{-3t} \\ e^{-3t} \end{pmatrix} \text{ and } \vec{X}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

are linearly indep. for any t.

then, the general soln. for (7.1) is given by

$$\begin{aligned} \vec{X}(t) &= C_1 \vec{X}^{(1)}(t) + C_2 \vec{X}^{(2)}(t) \\ &= C_1 \begin{pmatrix} -e^{-3t} \\ e^{-3t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -C_1 e^{-3t} + C_2 e^{-t} \\ C_1 e^{-3t} + C_2 e^{-t} \end{pmatrix} = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \end{aligned} \tag{9.1}$$

IVP:

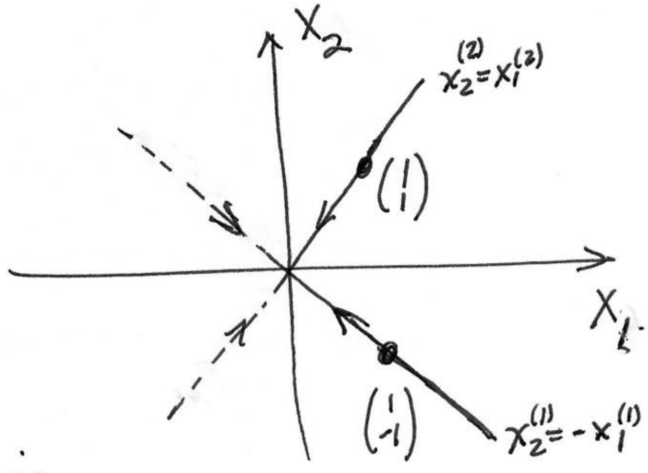
$$\begin{cases} \vec{X}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ \vec{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} \end{cases}$$

Subst. $t=0$ into (9.1)

$$X_1^{(1)}(0) = -c_1 e^0 + c_2 e^0 = 1$$
$$\Rightarrow$$
$$X_2^{(1)}(0) = c_1 e^0 + c_2 e^0 = -1$$

or $\begin{cases} -C_1 + C_2 = 1 \\ C_1 + C_2 = -1 \end{cases} \Rightarrow \begin{matrix} 2C_2 = 0 \Rightarrow \boxed{C_2 = 0} \\ \boxed{C_1 = -1} \end{matrix}$

$$\Rightarrow \quad \vec{X}^{(1)}(t) = \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix} = \begin{pmatrix} X_1^{(1)} \\ X_2^{(1)} \end{pmatrix} \Rightarrow \boxed{X_2^{(1)} = -X_1^{(1)}}$$



If I.C. is

$$\vec{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ then } \vec{X}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} X_1^{(2)} \\ X_2^{(2)} \end{pmatrix} \Rightarrow \boxed{X_2^{(2)} = X_1^{(2)}}$$

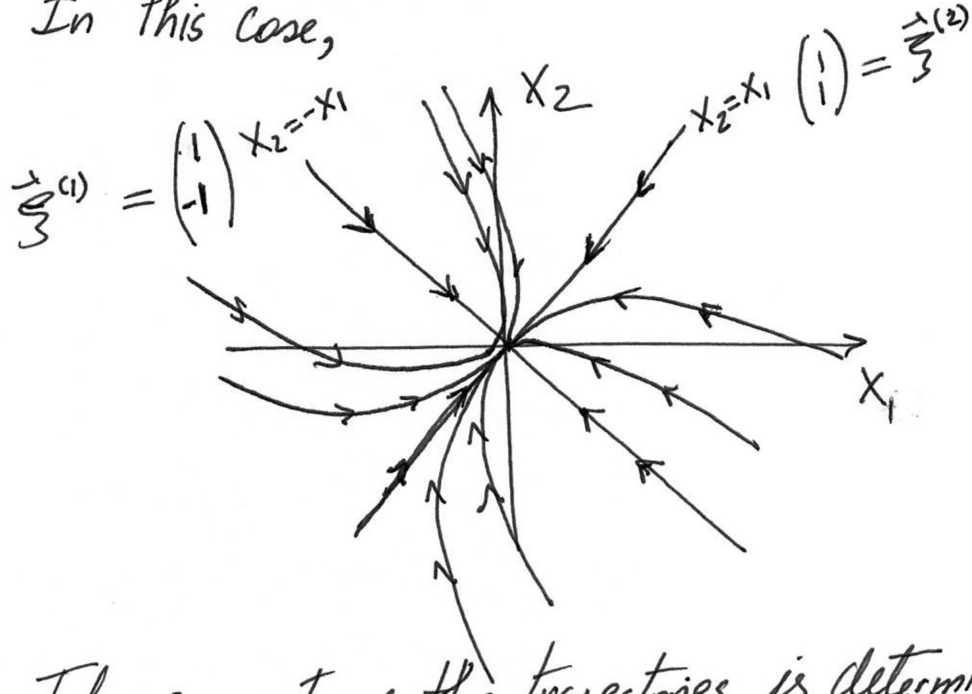
Notice that as $t \to \infty$

$$\vec{X}^{(1)}(t) \to 0, \quad \vec{X}^{(2)}(t) \to 0.$$

This gives direction of arrows in Phase Portrait.

By choosing many other ICs., we obtain many trajectories to form the phase portrait

In this case,



The concavity of the trajectories is determined by the direction in which the trajectories decay faster and decay slower.

In direction $\begin{pmatrix}1\\-1\end{pmatrix}$ the decay is given by e^{-3t}

In direction $\begin{pmatrix}1\\1\end{pmatrix}$ the decay is given by e^{-t}

Notice, that the only equilibrium solution (steady state)

$$X'(t) = A\vec{x}(t)$$
$$\overset{\|}{0}$$

$$\Rightarrow \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

Since $\det(A) \neq 0$ only soln is trivial soln $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

" origin in phase-plane

Classification of Equilibrium point and stability

For $$\vec{x}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The origin is called a <u>Node</u> and it is asymptotically stable because

$$\lim_{t\to\infty} \vec{x}(t) = 0, \text{ for any soln. } \vec{x}(t).$$

Origin is a Saddle point

$$\vec{X}'(t) = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (13.1)$$

Eeigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = 0$$

$$\text{or } (\lambda-1)^2 - 9 = \lambda^2 - 2\lambda + 1 - 9 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0 \Leftrightarrow (\lambda - 4)(\lambda + 2) = 0$$

$\Rightarrow \lambda_1 = 4, \quad \lambda_2 = -2$ are eigenvalues of A.

Eigenvectors:

$\lambda_1 = 4$

$$\begin{bmatrix} 1-4 & -3 \\ -3 & 1-4 \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -3\xi_1^{(1)} - 3\vec{\xi}_2^{(1)} = 0 \\ -3\xi_1^{(1)} - 3\xi_2^{(1)} = 0 \end{cases} \Rightarrow \vec{\xi}_2^{(1)} = -\xi^{(1)}$$

or $\quad \vec{\xi}^{(1)} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

and a solution of (13.1) is given by

$$\boxed{\vec{X}^{(1)}(t) = \vec{\xi}^{(1)} e^{4t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} = \begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix}}$$

$\underline{\underline{\lambda_2 = -2}}$

$$\begin{bmatrix} 1-(-2) & -3 \\ -3 & 1-(-2) \end{bmatrix} \begin{bmatrix} \vec{\xi}_1^{(2)} \\ \vec{\xi}_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3\vec{\xi}_1^{(2)} - 3\vec{\xi}_2^{(2)} = 0 \\ -3\vec{\xi}_1^{(2)} + 3\vec{\xi}_2^{(2)} = 0 \end{cases} \Rightarrow \xi_2^{(2)} = \xi_1^{(2)}$$

$$\Rightarrow \vec{\xi}^{(2)} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, another soln. of (13.1) is given by

$$\boxed{\vec{X}^{(2)}(t) = \vec{\xi}^{(2)} e^{-2t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}}$$

We can easily show that $\vec{X}^{(1)}(t)$ and $\vec{X}^{(2)}(t)$ are linearly indep for all t. In fact,

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \begin{vmatrix} e^{4t} & e^{-2t} \\ -e^{4t} & e^{-2t} \end{vmatrix} = e^{2t} + e^{2t} = 2e^{2t} \neq 0$$

for all t.

As a consequence, the general soln. is given by

$$\boxed{\vec{X}(t) = c_1 \vec{X}^{(1)}(t) + c_2 \vec{X}^{(2)}(t) = c_1 \begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} c_1 e^{4t} + c_2 e^{-2t} \\ -c_1 e^{4t} + c_2 e^{-2t} \end{pmatrix}}.$$

Phase Portrait

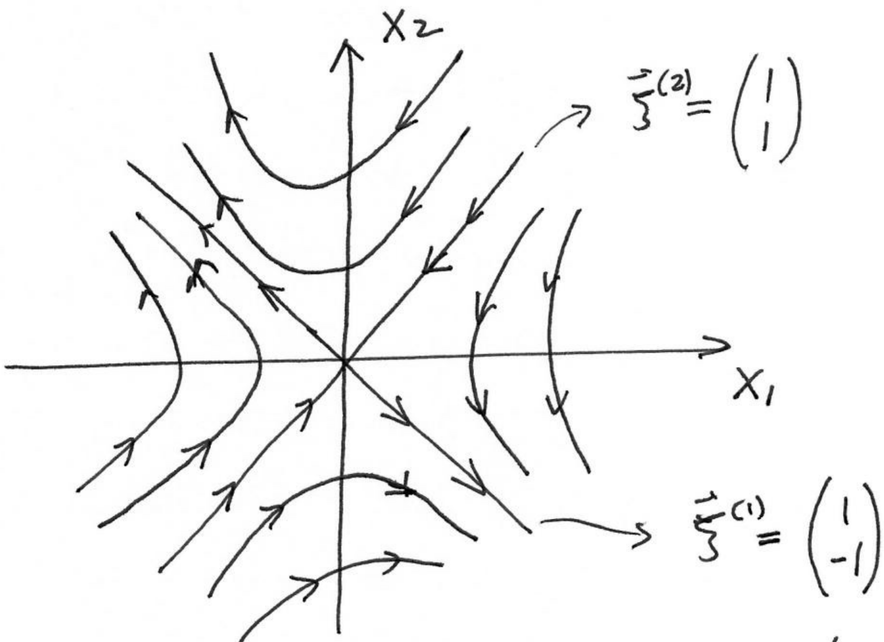
Let us graph some trajectories

- If $C_2 = 0 \Rightarrow \vec{x}(t) = C_1 \begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \boxed{x_2 = -x_1}$$

- If $C_1 = 0 \Rightarrow \vec{x}(t) = C_2 \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \boxed{x_2 = x_1}$$



Equil. point at origin is called a Saddle point

and $\lim_{t \to \infty} \vec{x}(t) = \begin{cases} \vec{0}, & C_1 = 0 \\ \begin{pmatrix} \infty \\ -\infty \end{pmatrix}, & C_1 > 0 \\ \vdots & \end{cases}$ unstable.