7.6 Complex Eigenvalues

Let us begin with an example

$$\vec{X}'(t) = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix} \vec{X} = A\vec{X}, \quad A = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix} \tag{1.1}$$

Again, we look for solutions in the form:

$$\vec{X}(t) = \vec{\xi} e^{rt} \tag{1.2}$$

Therefore, $\vec{\xi} r \cancel{e^{rt}} = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix} \vec{\xi} \cancel{e^{rt}}$

or $$\begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix} \vec{\xi} - r\vec{\xi} = 0$$

or $$\begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix} \vec{\xi} - rI\vec{\xi} = 0$$

or $$\begin{pmatrix} -1-r & 9 \\ -1 & -1-r \end{pmatrix} \vec{\xi} = 0 \Leftrightarrow (A - rI)\vec{\xi} = 0$$

For non-trivial solutions

$$|A - rI| = 0 \Leftrightarrow (-1-r)(-1-r) + 9 = 0$$

$$(r+1)^2 + 9 = 0 \Leftrightarrow r^2 + 2r + 10 = 0$$

Then, $$r = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm \frac{\sqrt{-36}}{2} i = -1 \pm 3i$$

So eigenvalues are $r_1 = -1+3i$

$r_2 = -1-3i$

Eigenvectors for $\boxed{r_1 = -1+3i}$

are obtained by solving

$$\begin{pmatrix} -1-(-1+3i) & 9 \\ -1 & -1-(-1+3i) \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $$\begin{pmatrix} -3i & 9 \\ -1 & -3i \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we arrive to the system of equations

$$\begin{cases} -3i\,\xi_1^{(1)} + 9\,\xi_2^{(1)} = 0 & (2.1) \\ -\xi_1^{(1)} - 3i\,\xi_2^{(1)} = 0 & (2.2) \end{cases}$$

Clearly, equ. (2.1) is a multiple of equ. (2.2). In fact,

$$3i\,(\text{Equ. } 2) = \text{Equ. } (2.1)$$

This is expected. otherwise, it would not non-trivial Solutions.

Then, Solutions of System (2.1)-(2.2) are

$$\xi_1^{(1)} = -3i\,\xi_2^{(1)}$$

or $$\vec{\xi}^{(1)} = c\begin{pmatrix} -3i \\ 1 \end{pmatrix}$$

and a Complex Solution of (1.1) given by (1.2) is

$$\vec{X}^{(1)}(t) = e^{(-1+3i)t}\begin{pmatrix} -3i \\ 1 \end{pmatrix}$$

$$= e^{-t}e^{3it}\begin{pmatrix} -3i \\ 1 \end{pmatrix}$$

$$= e^{-t}(\cos 3t + i\sin 3t)\begin{pmatrix} -3i \\ 1 \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} +3\sin 3t - i\,3\cos 3t \\ \cos 3t + i\sin 3t \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{-t}\sin 3t - i\,3e^{-t}\cos 3t \\ e^{-t}\cos 3t + i e^{-t}\sin 3t \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{-t}\sin 3t \\ e^{-t}\cos 3t \end{pmatrix} + i\begin{pmatrix} -3e^{-t}\cos 3t \\ e^{-t}\sin 3t \end{pmatrix}$$

$$= \vec{u}(t) + i\vec{v}(t)$$

We can easily show that $u(t)$ and $v(t)$ are solutions of (1.1).

$u(t)$: real part of $\vec{x}$; $v(t)$: Imaginary part of $\vec{x}$

Thm 7.4.5

Consider $\vec{x}'(t) = A\vec{x}(t)$, A $n \times n$ matrix of real entries (4.1)

If $\vec{x}(t) = \vec{u}(t) + i\vec{v}(t)$ (4.2)

is a complex-valued solution of the above system then its real part $\vec{u}(t)$ and its imaginary part $v(t)$ are also solutions of this System.

Proof.

Subst. (4.2) into (4.1)

$$\left(\vec{u}(t) + i\vec{v}(t)\right)' = A\left(\vec{u}(t) + i\vec{v}(t)\right)$$

$$\Rightarrow \left(\vec{u}'(t) - A\vec{u}(t)\right) + i\left(\vec{v}'(t) - A\vec{v}(t)\right) = \vec{0}$$

$$\Rightarrow \quad \vec{u}'(t) - A\vec{u}(t) = 0 \Rightarrow \boxed{\vec{u}'(t) = A\vec{u}(t)} \checkmark$$

$$\text{and} \quad \vec{v}'(t) - A\vec{v}(t) = 0 \Rightarrow \boxed{\vec{v}'(t) = A\vec{v}(t)}$$

7.6 Complex eigenvalues. (cont.)

If eigenvalues are pure imaginary numbers:

$\lambda_1 = i\beta$, $\lambda_2 = -i\beta$, then the equilibrium point $\vec{x} = \vec{0}$ is called a center.

$$\vec{x}'(t) = \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix} \vec{x}(t)$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ -5 & -3-\lambda \end{vmatrix} = \begin{vmatrix} -(\lambda-3) & 2 \\ -5 & -(\lambda+3) \end{vmatrix} = (\lambda-3)(\lambda+3) + 10$$

$$= \lambda^2 - 9 + 10 = \lambda^2 + 1 = 0$$

$\Rightarrow \lambda_1 = i, \quad \lambda_2 = -i$

Eigenvectors:

$\lambda_1 = i$

$$\begin{pmatrix} 3-i & 2 \\ -5 & -3-i \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} (3-i)\xi_1' + 2\xi_2' \\ -5\xi_1' + (-3-i)\xi_2' \end{pmatrix} \stackrel{\vec{0}}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[(3-i)\xi_1' + 2\xi_2' = 0\right] * -\left(\frac{3+i}{2}\right) \quad \text{lin. dep.}$$

$$-5\xi_1' - (3+i)\xi_2' = 0$$

Solving the first equ. for ξ_2' :

$$\xi_2' = -\frac{3-i}{2}\xi_1'$$

$$\Rightarrow \quad \vec{\xi}^{(1)} = \begin{pmatrix} 2 \\ -3+i \end{pmatrix}$$

$\Rightarrow$ Complex solution:

$$z^{(1)}(t) = e^{it}\begin{pmatrix} 2 \\ -3+i \end{pmatrix} = \begin{pmatrix} 2\cos t + 2i\sin t \\ (-3+i)(\cos t + i\sin t) \end{pmatrix} =$$

$$= \begin{pmatrix} 2\cos t + 2i\sin t \\ -3\cos t - \sin t + i(\cos t - 3\sin t) \end{pmatrix} =$$

$$= \begin{pmatrix} 2\cos t \\ -3\cos t - \sin t \end{pmatrix} + i\begin{pmatrix} 2\sin t \\ \cos t - 3\sin t \end{pmatrix} = u(t) + i\,v(t)$$

$u(t)$, and $v(t)$ are real Solus and they are l. indep.

In fact,

$$\begin{vmatrix} 2\cos t & 2\sin t \\ -3\cos t - \sin t & \cos t - 3\sin t \end{vmatrix} = 2\cos^2 t - \cancel{6\cos t\sin t} + \cancel{6\sin t\cos t} + 2\sin^2 t = 2 \neq 0.$$

therefore, General Soln.:

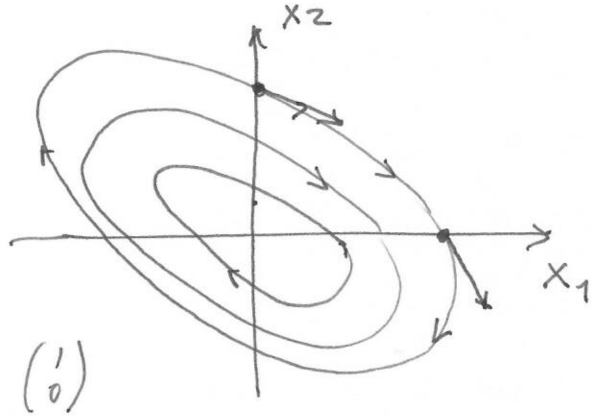
$$\vec{X}(t) = C_1\begin{pmatrix} 2\cos t \\ -3\cos t - \sin t \end{pmatrix} + C_2\begin{pmatrix} 2\sin t \\ \cos t - 3\sin t \end{pmatrix}$$

and

$$x_1(t) = 2c_1 \cos t + 2c_2 \sin t.$$

$$x_2(t) = (-3c_1 + c_2)\cos t + (-c_1 - 3c_2)\sin t$$

Permanent oscilations in both directions.



Center
stable.

For $\hat{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\hat{x}'(0) = \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

For $\hat{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$