7.9 Nonhomogeneous Linear Systems.

We will discuss two different methods to solve nonhomogeneous linear systems of ODE's.

i) Diagonalization method.

ii) Method of undetermined coefficients.

Diagonalization.

Start by considering the difference between

$$\vec{x}' = \begin{pmatrix} -5/4 & 3/4 \\ 3/4 & -5/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}$$

or
$$\begin{cases} x_1' = -\frac{5}{4}x_1 + \frac{3}{4}x_2 + 2t \\ x_2' = \frac{3}{4}x_1 - \frac{5}{4}x_2 + e^t \end{cases}$$

and
$$y' = \begin{pmatrix} -1/2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} t + \frac{e^t}{2} \\ t - \frac{e^t}{2} \end{pmatrix}$$

or
$$\begin{cases} y_1' = -\frac{1}{2}y_1 + t + e^t/2 \\ y_2' = -2y_2 + t - \frac{e^t}{2} \end{cases}$$

7.9 #9) $\tilde{X}' = \begin{pmatrix} -5/4 & 3/4 \\ 3/4 & -5/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}$

(I) Diagonalization Method

a) Eigenvalues: $\begin{vmatrix} -5/4-\lambda & 3/4 \\ 3/4 & -5/4-\lambda \end{vmatrix} = \left(\lambda + 5/4\right)^2 - \frac{9}{16} = 0$

$$\lambda^2 + \frac{5}{2}\lambda + \frac{25}{16} - \frac{9}{16} = 0 \Leftrightarrow \lambda^2 + \frac{5}{2}\lambda + 1$$

$$\lambda_{1/2} = \frac{-5/2 \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{-5/2 \pm 3/2}{2} = \begin{cases} -1/2 \\ -4/2 = -2 \end{cases}$$

b) Eigenvectors:

$\boxed{\lambda_1 = -1/2}$

$$\begin{pmatrix} -5/4 + 1/2 & 3/4 \\ 3/4 & -5/4 + 1/2 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi_1^{(1)} = \xi_2^{(1)} \quad \boxed{\tilde{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\boxed{\lambda_2 = -2}$

$$\begin{pmatrix} -5/4 + 8/4 & 3/4 \\ 3/4 & 3/4 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi_1^{(2)} = -\xi_2^{(2)} \quad \boxed{\tilde{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

c) <u>Matrix T and change of Variables</u>
<u>Diagonal System</u>

$$\vec{x} = T\vec{y} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ y_1 - y_2 \end{pmatrix}$$

New diagonal system

$$\vec{x}' = A\vec{x} + \vec{g} \Leftrightarrow T\vec{y}' = AT\vec{y} + \vec{g}$$

$$\Leftrightarrow \vec{y}' = \underbrace{T^{-1}AT}_{D}\vec{y} + T^{-1}\vec{g}$$

or $$\vec{y}' = \begin{pmatrix} -1/2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + T^{-1}\begin{pmatrix} 2t \\ e^t \end{pmatrix}$$

$$T^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{y}' = \begin{pmatrix} -1/2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2t \\ e^t \end{pmatrix}}$$

or $$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -1/2\, y_1 + \frac{1}{2}(2t + e^t) \\ -2y_2 + \frac{1}{2}(2t - e^t) \end{pmatrix}$$

d) Two ODE's: Solving

$$\begin{cases} y_1' + \frac{1}{2} y_1 = t + \frac{1}{2} e^t \\ y_2' + 2 y_2 = t - \frac{1}{2} e^t \end{cases}$$

$$y_1 e^{t/2} = \int t e^{t/2} dt + \frac{1}{2} \int e^{3/2 t} dt + C_1$$

$$\Rightarrow \quad y_1(t) = e^{-t/2} \left[2t e^{t/2} - 4 e^{t/2} + \frac{1}{2} \frac{2}{3} e^{3/2 t} + C_1 \right]$$

$$\Rightarrow \quad \boxed{y_1(t) = 2t - 4 + \frac{1}{3} e^t + C_1 e^{-t/2}}$$

$$y_2(t) = e^{-2t} \left[\int t e^{2t} dt - \frac{1}{2} \int e^{3t} dt + C_2 \right]$$

$$\Rightarrow y_2(t) = e^{-2t} \left[\frac{t}{2} e^{2t} - \frac{1}{4} e^{2t} - \frac{1}{6} e^{3t} + C_2 \right]$$

or

$$\boxed{y_2(t) = \frac{t}{2} - \frac{1}{4} - \frac{1}{6} e^t + C_2 e^{-2t}}$$

$$\int \underset{f}{t}\, \underset{g'}{e^{t/2}} dt = t\, 2 e^{t/2} - \int 2 e^{t/2} dt = 2t e^{t/2} - 4 e^{t/2}$$

$$\int t e^{2t} dt = \frac{1}{2} t e^{2t} - \int \frac{e^{2t}}{2} dt = \frac{t}{2} e^{2t} - \frac{1}{4} e^{2t}$$

e) <u>Back to original Variables:</u> $\vec{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \overset{T}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \overset{\vec{y}}{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}} = \begin{pmatrix} y_1 + y_2 \\ y_1 - y_2 \end{pmatrix}$$

or
$$\vec{x} = \begin{pmatrix} \frac{5}{2}t - 17/4 + \frac{1}{6}e^{t} + c_1 e^{-t/2} + c_2 e^{-2t} \\ \frac{3}{2}t - 15/4 + \frac{1}{2}e^{t} + c_1 e^{-t/2} - c_2 e^{-2t} \end{pmatrix}$$

or
$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix} e^{t} - \begin{pmatrix} 17/4 \\ 15/4 \end{pmatrix}$$

7.9 # 9) Cont.

(II) Undetermined coefficients:

$$\vec{g}(t) = \begin{pmatrix} 2t \\ e^t \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

a) Particular soln:

$$\vec{X}_p(t) = \vec{a}t + \vec{b}e^t + \vec{c} \quad \vec{a}, \vec{b}, \vec{c} = ?$$

$$\vec{X}_p'(t) = \vec{a} + \vec{b}e^t = A\vec{x}_p(t) + \begin{pmatrix} 2t \\ e^t \end{pmatrix}$$

$$= A\vec{a}t + A\vec{b}e^t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t + A\vec{c}$$

$$\Rightarrow \boxed{A\vec{a} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}}, \quad \boxed{A\vec{c} = \vec{a}}$$

$$A\vec{b} - \vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Leftrightarrow \boxed{(A - I)\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}}$$

b) Solving:

$$\begin{pmatrix} -5/4 & 3/4 \\ 3/4 & -5/4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\frac{3}{4} a_1 = \frac{5}{4} a_2 \Rightarrow \boxed{a_1 = 5/3\, a_2}$$

$$-5/4\, a_1 + 3/4\, a_2 = 2 \Rightarrow \frac{-25}{12} a_2 + \frac{3}{4} a_2 = -2$$

$$a_2 \left(\frac{-25 + 9}{12} \right) = -2 \Rightarrow a_2 = -2 \left(\frac{\cancel{12}^{\,3}}{\cancel{-16}_{\,-4}} \right)$$

$$\boxed{a_2 = +\frac{3}{2}}$$

$$\Rightarrow \boxed{a_1 = \frac{5}{3}\left(\frac{3}{2}\right) = 5/2}$$

Then, $\boxed{\vec{a} = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}}$

Next equ.:

$$A\vec{c} = \vec{a} = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$$

$$\begin{pmatrix} -5/4 & 3/4 \\ 3/4 & -5/4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$$

$$\begin{pmatrix} -5/4 & 3/4 & 5/2 \\ 3/4 & -5/4 & 3/2 \end{pmatrix} \sim \begin{pmatrix} -1 & 3/5 & 2 \\ 0 & -\cancel{16}^{4}/\cancel{20}_{5} & 3 \end{pmatrix}^{3/4} \qquad 9/20$$

$$-\frac{4}{5}c_2 = 3 \Rightarrow \boxed{c_2 = -\frac{15}{4}}$$

$$-c_1 = 2 - \frac{3}{5}c_2 \Rightarrow c_1 = -2 - \frac{\cancel{3}}{\cancel{5}_1}\left(\frac{\cancel{15}^{3}}{4}\right)$$

$$c_1 = -\frac{8}{4} - 9/4 = -\frac{17}{4}$$

$$\Rightarrow \boxed{\vec{C} = \begin{pmatrix} -17/4 \\ -15/4 \end{pmatrix}}$$

Next. Equ.:

$$(A - I)\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -5/4-1 & 3/4 \\ 3/4 & -5/4-1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -9/4 & 3/4 \\ 3/4 & -9/4 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} -9b_1 + 3b_2 = 0 \\ 3/4\, b_1 - 9/4\, b_2 = -1 \end{cases}$$

$$3b_1 - 9b_2 = -4$$

$$\Rightarrow b_2 = 3b_1 \Rightarrow 3b_1 - 27b_1 = -4$$

$$\Rightarrow -24b_1 = -4 \Rightarrow \boxed{b_1 = \frac{1}{6}} \Rightarrow \boxed{b_2 = \frac{1}{2}}$$

Then, $\boxed{\breve{b} = \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}}$

Then, $\breve{X}_p(t) = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix} e^t - \begin{pmatrix} 17/4 \\ 15/4 \end{pmatrix}$

<u>General solution:</u>

$$\boxed{\breve{X}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix} e^t - \begin{pmatrix} 17/4 \\ 15/4 \end{pmatrix}}$$

7.9 Example 1

$$\vec{x}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

or $\quad \vec{x}'(t) = A\vec{x}(t) + \vec{g}(t)$

Eigenvalues of A

$$|A - rI| = 0 \iff \begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = 0 \iff (r+2)^2 - 1 = 0$$
$$\iff r^2 + 4r + 3 = 0$$
$$(r+3)(r+1) = 0$$

Two eigenvalues are

$r_1 = -3, \quad r_2 = -1$

Eigenvectors for $r_1 = -3$

$$\begin{pmatrix} -2+3 & 1 \\ 1 & -2+3 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$+\xi_1^{(1)} + \xi_2^{(1)} = 0 \Rightarrow \xi_1^{(1)} = -\xi_2^{(1)}$$
$$\xi_1^{(1)} + \xi_2^{(1)} = 0$$

$$\Rightarrow \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenvectors for $r_2 = -1$

$$\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \left(\vec{\xi}^{(2)} \right) = \vec{0}.$$

$$\xi_1^{(2)} = \xi_2^{(2)} \Rightarrow \vec{\xi}^{2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Two solns. for homogeneous associated system:

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

$$\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Calling $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

and introducing $\vec{y}$: $\vec{x} = T\vec{y}$

then,
$$\vec{x}'(t) = A\vec{x}(t) + \vec{g}(t)$$

reduces to
$$T\vec{y}'(t) = AT\vec{y}(t) + \vec{g}(t)$$

multiplying by T^{-1}:
$$\vec{y}'(t) = T^{-1}AT\vec{y}(t) + T^{-1}\vec{g}(t).$$

or
$$\vec{y}'(t) = D\vec{y}(t) + T^{-1}\vec{g}(t)$$

In our case,
$$\vec{y}'(t) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

or
$$\vec{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} -3y_1 + \frac{1}{2}(2e^{-t} - 3t) \\ -y_2 + \frac{1}{2}(2e^{-t} + 3t) \end{pmatrix}$$

Two Scalar linear 1^{st} order **ODE** completely uncoupled.

$$\begin{cases} y_1'(t) + 3y_1 = e^{-t} - \frac{3}{2}t & (3.1) \\ y_2' + y_2 = e^{-t} + \frac{3}{2}t & (3.2) \end{cases}$$

Solving (3.1)

$$IF = e^{\int 3dt} = e^{3t}$$

$$\frac{d}{dt}\left[y_1 e^{3t}\right] = e^{2t} - \frac{3}{2} t e^{3t}$$

$\int dt$:

$$y_1(t)\, e^{3t} = \frac{1}{2}e^{2t} - \frac{3}{2}\int t e^{3t} dt + C_1$$

or

$$y_1(t) = \frac{1}{e^{3t}}\left[\frac{1}{2}e^{2t} - \frac{3}{2}\left(\frac{t}{3}e^{3t} - \frac{1}{9}e^{3t}\right) + C_1\right]$$

or

$$\boxed{y_1(t) = \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + C_1 e^{-3t}}$$

Similarly, $\frac{d}{dt}\left[y_2(t)e^{t}\right] = 1 + \frac{3}{2}te^{t} \Rightarrow y_2(t)e^{t} = t + \frac{3}{2}\left(te^{t} - e^{t}\right) + C_2$

$$\Rightarrow \boxed{y_2(t) = te^{-t} + \frac{3}{2}t - \frac{3}{2} + C_2 e^{-t}}$$

$$\int \underset{\substack{\downarrow \\ f}}{t}\, \underset{\substack{\downarrow \\ g'}}{e^{3t}} dt = t\frac{1}{3}e^{3t} - \frac{1}{3}\int e^{3t}dt = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t}$$

$$\int te^{t}dt = te^{t} - \int e^{t}dt = te^{t} - e^{t}$$

Since $\vec{x} = T\vec{y}$

Then, $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + c_1 e^{-3t} + te^{-t} + \frac{3}{2}t - \frac{3}{2} + c_2 e^{-t} \\ -\frac{1}{2}e^{-t} + \frac{t}{2} - \frac{1}{6} - c_1 e^{-3t} + te^{-t} + \frac{3}{2}t - \frac{3}{2} + c_2 e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} (c_2 + \frac{1}{2})e^{-t} + t - \frac{4}{3} + c_1 e^{-3t} + te^{-t} \\ (c_2 - \frac{1}{2})e^{-t} + 2t - \frac{5}{3} - c_1 e^{-3t} + te^{-t} \end{pmatrix}$$

or

$$\boxed{\vec{x}(t) = \underbrace{c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}}_{\text{General Soln. of Associated homogeneous system.}} + \underbrace{\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}}_{\text{particular Soln.}}}$$

Method of Undetermined Coefficients

$$\vec{X}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = A\vec{x} + \vec{g}(t) \qquad (5.1)$$

$$\vec{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$

Proposed particular Soln:

$$\vec{X}_p(t) = \vec{a}te^{-t} + \vec{b}e^{-t} + \vec{c}t + \vec{d} \qquad (5.2)$$

Subst. (5.2) into (5.1)

$$\vec{X}_p'(t) = \vec{a}e^{-t} - \vec{a}te^{-t} - \vec{b}e^{-t} + \vec{c} =$$

$$= A\vec{a}te^{-t} + A\vec{b}e^{-t} + A\vec{c}t + A\vec{d} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}t$$

$$\Rightarrow \quad (A\vec{a} + \vec{a})te^{-t} + (A\vec{b} - (\vec{a} - \vec{b}))e^{-t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}e^{-t}$$

$$+ (A\vec{d} - \vec{c}) + \left(A\vec{c} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)t = \vec{0} + \vec{0}t + \vec{0}te^{-t} + \vec{0}e^{-t}$$

$$\Rightarrow \quad A\vec{a} = -\vec{a} \quad \Rightarrow \quad \boxed{\vec{a} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \ \boxed{r = -1} \text{ eigenvalue.}$$

$$A\vec{b} = \vec{a} - \vec{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$A\vec{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$A\vec{d} = \vec{c}$$

$$A\vec{b} = \vec{a} - \vec{b} - \begin{pmatrix}2\\0\end{pmatrix} = \overset{\vec{a}}{\overset{\|}{\begin{pmatrix}\alpha\\\alpha\end{pmatrix}}} - \begin{pmatrix}b_1\\b_2\end{pmatrix} - \begin{pmatrix}2\\0\end{pmatrix}$$

$$A\vec{b} = \begin{pmatrix}\alpha - b_1 - 2\\ \alpha - b_2 + 0\end{pmatrix} =$$

$$(A + I)\vec{b} = \begin{pmatrix}\alpha - 2\\ \alpha\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\vec{b} = \begin{pmatrix}\alpha - 2\\ \alpha\end{pmatrix}$$

$$\left(\begin{array}{cc|c}-1 & 1 & \alpha - 2\\ 1 & -1 & \alpha\end{array}\right) \sim \begin{pmatrix}-1 & 1 & \alpha - 2\\ 0 & 0 & \alpha + \alpha - 2\end{pmatrix}$$

Only possibility $2\alpha - 2 = 0 \Rightarrow \boxed{\alpha = 1}$
for soln $\neq 0$, Then, $\boxed{\vec{a} = \begin{pmatrix}1\\1\end{pmatrix}}$

So
$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = \overset{=\vec{a}}{\begin{pmatrix}1\\1\end{pmatrix}} - \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

$$\Rightarrow -b_1 + b_2 = -1 \Rightarrow \boxed{b_2 = b_1 - 1}$$

$$b_1 - b_2 = 1 \qquad \vec{b} = \begin{pmatrix}b_1\\ b_1 - 1\end{pmatrix} = b_1\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix}$$

or $\boxed{\vec{b} = \begin{pmatrix}0\\-1\end{pmatrix},}$ for $b_1 = 0$.

Also, from

$$A\vec{C} = -\begin{pmatrix}0\\3\end{pmatrix} \Leftrightarrow \begin{pmatrix}-2 & 1 & 0\\1 & -2 & -3\end{pmatrix} \sim \begin{pmatrix}-2 & 1 & 0\\0 & -3 & -6\end{pmatrix}$$

$$-3c_2 = -6 \Rightarrow \boxed{c_2 = 2}$$

$$\Rightarrow -2c_1 = -2 \Rightarrow \boxed{c_1 = 1}$$

or $\boxed{\vec{c} = \begin{pmatrix}1\\2\end{pmatrix}}$

and $A\vec{d} = \vec{C} \Leftrightarrow \begin{pmatrix}-2 & 1\\1 & -2\end{pmatrix}\begin{pmatrix}d_1\\d_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

or $\begin{pmatrix}-2 & 1 & 1\\1 & -2 & 2\end{pmatrix} \sim \begin{pmatrix}-2 & 1 & 1\\0 & -3 & 5\end{pmatrix} \Rightarrow -3d_2 = 5 \Rightarrow \boxed{d_2 = -5/3}$

$$\Rightarrow d_1 = -\frac{1}{2}(1-d_2)$$

or $d_1 = -\frac{1}{2}\left(1+\frac{5}{3}\right) = -\frac{4}{3}$

Therefore, $\boxed{\vec{d} = \begin{pmatrix}-5/3\\-4/3\end{pmatrix}}$

and the particular soln. is given by

$$\vec{X}_p(t) = \begin{pmatrix}1\\1\end{pmatrix}te^{-t} + \begin{pmatrix}0\\-1\end{pmatrix}e^{-t} + \begin{pmatrix}1\\2\end{pmatrix}t + \begin{pmatrix}-5/3\\-4/3\end{pmatrix} \checkmark$$