Summarizing

Different Equations and techniques to Solve them using Power Series.

1) No Singular points. All ordinary points.

$$y'' - xy' - y = 0$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!} x^{2n+1}$$

2) Legendre equation
Singular points at $x = \pm 1$

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

Power series Soln. around $x=0$ "ordinary point"

In book

3) Analytic functions (not polynomials as coefficients)

$$e^x y'' + xy = 0$$

All points are ordinary points.

See Notes

4) $\cos x \, y'' + xy' - 2y = 0$

Singularity at $x = \frac{\pi}{2}$

So no problem to find power series

Soln. around $x = 0$.

5.3 #12) $e^x y'' + xy = 0.$ (Very good!)

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0.$$

or

$$\sum_{2} n(n-1) a_n x^{n-2} + \sum_{n=2} n(n-1) a_n x^{n-1} + \sum_{n=2} \frac{n(n-1)}{2} a_n x^n + \sum_{2} \frac{n(n-1)}{6} a_n x^{n+1} +$$

$$+ \sum_{2} \frac{n(n-1)}{12} a_n x^{n+2} + \sum_{2} \frac{n(n-1)}{120} a_n x^{n+3} + \sum_{n=0} a_n x^{n+1} = 0$$

or

$$\sum_{0} (n+2)(n+1) a_{n+2} x^n + \sum_{1} (n+1) n a_{n+1} x^n + \sum_{2} \frac{n(n-1)}{2} a_n x^n + \sum_{3} \frac{(n-1)(n-2)}{6} a_{n-1} x^n$$

$$+ \sum_{4} \frac{(n-2)(n-3)}{12} a_{n-2} x^n + \sum_{5} \frac{(n-3)(n-4)}{120} a_{n-3} x^n + \sum_{n=1} a_{n-1} x^n = 0.$$

x^0: $2(1) a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

x^1: $3(2) a_3 + 2(1) \cancel{a_2}^{0} + a_0 = 0$
$\Rightarrow \boxed{a_3 = -\frac{a_0}{6}}$

x^2: $4(3) a_4 + 3(2) a_3 + \frac{2(1)}{2} \cancel{a_2}^{0} + a_1 = 0$
$12 a_4 + 6 a_3 + a_1 = 0 \Rightarrow a_4 = \frac{1}{12}(-a_1 - 6a_3)$

$\Rightarrow \boxed{a_4 = \frac{1}{12}(a_0 - a_1)}$

x^3: $5(4)a_5 + 4(3)a_4 + \frac{3(2)}{2}a_3 + \left(\frac{2(1)}{6}+1\right)a_2$

$$\Rightarrow a_5 = \frac{-1}{20}\left[12a_4 + 3a_3 + \frac{4}{3}\cancel{a_2}^{\,0}\right]$$

$$\Rightarrow a_5 = -\frac{1}{20}\left[a_0 - a_1 - \frac{a_0}{2}\right] = -\frac{1}{20}\left[\frac{a_0}{2} - a_1\right]$$

so $\boxed{a_5 = -\frac{1}{20}\left[\frac{a_0}{2} - a_1\right]}$

Therefore,

$$y(x) = a_0 + a_1 x - \frac{a_0}{6}x^3 + \frac{1}{12}(a_0 - a_1)x^4 - \frac{1}{20}\left(\frac{a_0}{2} - a_1\right)x^5 + \cdots$$

or $$y(x) = a_0\left[1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \cdots\right] +$$

$$+ a_1\left[x - \frac{x^4}{12} + \frac{x^5}{20} \cdots\cdots\right]$$

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x^4: $6(5)a_6 + 5(4)a_5 + \frac{4(3)}{2}a_4 + \left[\frac{3(2)}{6} + 1\right]a_3 +$

$$+ \frac{2(1)}{12}\cancel{a_2}^{\,0} = 0.$$

$$a_6 = \frac{-1}{30}\left[20a_5 + 6a_4 + 2a_3\right] = \frac{-1}{30}\left[a_1 - \cancel{\frac{a_0}{2}} + \cancel{\frac{1}{2}a_0} - \frac{1}{2}a_1 - \frac{a_0}{3}\right] = -\frac{a_1}{60} + \frac{a_0}{90}$$

$$\Rightarrow \boxed{a_6 = -\frac{a_1}{60} + \frac{a_0}{90}}$$

$$\boxed{a_6 = -\frac{a_1}{60} + \frac{a_0}{90}}$$

Therefore,

$$y(x) = a_0\left[1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \frac{x^6}{90} + \cdots\right]$$

$$+ a_1\left[x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots\right]$$