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Name:____________________________

Section: ____________________________

Instructor:____________________________

# Math 303 (Engineering Mathematics II)

# Quiz 4

Winter 2018 **TIME LIMIT 2.5 HOURS**

- Answer all question and show all your work to receive the full credit.
- Work on scratch paper **will not** be graded. Use the back on the left if you need more space.
- Basic scientific calculator or testing center calculator are allowed (not graphing or symbolic ones).

**For instructor use only:**

| # | Possible | Earned |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 20 | |
| Total | 40 | |

## Integrals including *sin* and *cos*

1. $\int \sin ax \, dx = -\frac{1}{a}\cos ax + c$

2. $\int \cos ax \, dx = \frac{1}{a}\sin ax + c$

3. $\int x \sin ax \, dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + c$

4. $\int x \cos ax \, dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + c$

1. a) Solve the given eigenvalue problem: find the values of the parameter $\lambda$ (eigenvalues) for which the given boundary value problem has non-zero solutions (eigenfunctions). Assuming that $\lambda$ is real, consider all possible cases (positive, negative, and zero). Also, find the corresponding eigenfunctions:

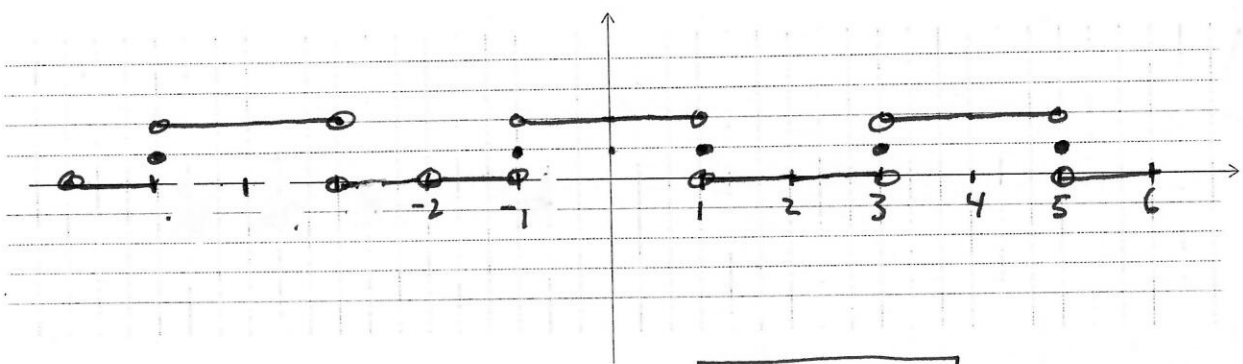
$$X' + \lambda X = 0, \qquad X'(0) = 0,\ X'(L) = 0$$

b) Use the eigenfunctions found in part (a) to construct a Fourier series expansion of the function

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}.$$

Why is it possible?

d) Sketch the function to which the Fourier series found in part (b) converges. Sketch it for three periods.




a) $\boxed{X'' + \lambda X = 0,}$ Charact. polyn: $\boxed{r^2 + \lambda = 0}$

i) $\lambda > 0$, $r_{1,2} = \pm\sqrt{\lambda}\, i \Rightarrow X(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$

$X'(x) = -\sqrt{\lambda}\, C_1 \sin\sqrt{\lambda}\,x + \sqrt{\lambda}\, C_2 \cos\sqrt{\lambda}\,x$

Then, $X'(0) = 0 \Rightarrow \sqrt{\lambda}\, C_2 \overset{=1}{\cos(0)} = 0 \Rightarrow \boxed{C_2 = 0}$ because $\sqrt{\lambda} > 0$.

$\Rightarrow X(x) = C_1 \cos\sqrt{\lambda}\,x$

$\Rightarrow X'(L) = 0 \Rightarrow -\sqrt{\lambda}\, C_1 \sin\sqrt{\lambda}\,L = 0 \Rightarrow \sin\sqrt{\lambda}\,L = 0$

otherwise $C_1 = 0$ (trivial soln.) b/c $\sqrt{\lambda} > 0$

$\Rightarrow \sqrt{\lambda}\,L = n\pi \Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$ eigenvalues $n = 1, 2, \ldots$ Turn

ii) $\lambda = 0$, $X''(x) = 0 \Rightarrow X(x) = C_1 x + C_2$

$X'(x) = C_1$, $\quad 0 = X'(0) = C_1 \Rightarrow \boxed{C_1 = 0}$

Also, $0 = X'(L) = C_1$. Therefore $X_0(x) = C_2 = C_2(1)$

or $\boxed{X_0(x) = 1}$ Corresponding eigenfn. to $\lambda_0 = 0$.

The Corresponding eigenfunctions are: for $\lambda_n$; $n=1,2\ldots$

$$X_n(x) = \cos\sqrt{\lambda}\,x = \cos\frac{n\pi}{L}x, \quad n=1,2,\ldots$$

and $X_0(x)=1$, for $\lambda_0=0$

iii) $\lambda<0$, $\quad r_{1,2} = \pm\sqrt{-\lambda}$ real.

$$\Rightarrow X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$$

or $$X(x) = C_1\cosh(\sqrt{-\lambda}x) + C_2\sinh(\sqrt{-\lambda}x).$$

$$X'(x) = \sqrt{-\lambda}\,C_1\sinh(\sqrt{-\lambda}x) + \sqrt{-\lambda}\,C_2\cosh(\sqrt{-\lambda}x).$$

$$X'(0)=0 \Rightarrow \sqrt{-\lambda}\,C_2\underbrace{\cosh(0)}_{=1}=0 \Rightarrow \boxed{C_2=0}, \text{ b/c } \lambda\neq0.$$

Also $$X'(L)=0 \Rightarrow \underbrace{\sqrt{-\lambda}}_{\neq0}\,C_1\sinh\underbrace{(\sqrt{-\lambda}L)}_{\neq0} = 0 \Rightarrow \boxed{C_1=0}$$

only trivial soln.

then $\lambda<0$ not an eigenvalue.

<u>Fourier Cosine Series:</u>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{2}x, \qquad f(x) = \begin{cases}1, & 0\le x\le 1\\ 0, & 1\le x\le 2\end{cases}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 dx = 1, \quad a_n = \frac{2}{2}\int_0^1 \cos\frac{n\pi}{2}x\,dx$$

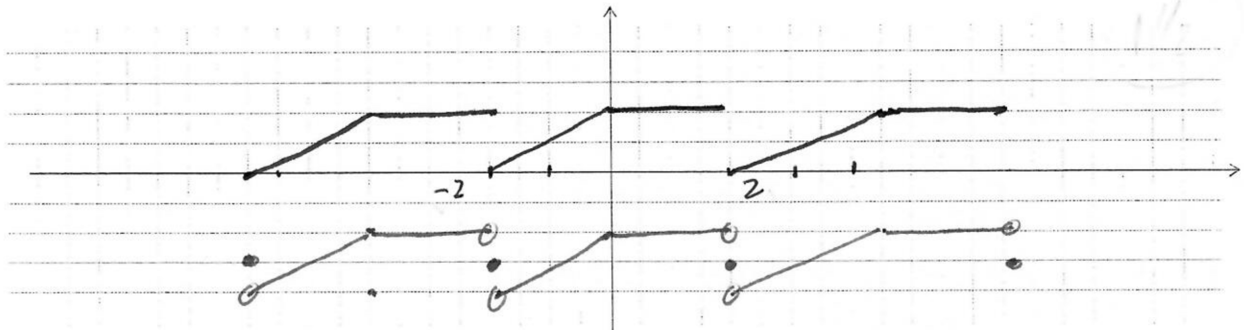
$$= \frac{L}{n\pi}\sin\frac{n\pi}{2}x\Big|_0^1 = \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)-0\right] = \begin{cases}\frac{2}{n\pi}, & 1,5,9,13,\ldots\\ -\frac{2}{n\pi}, & n=3,7,11,\ldots\\ 0, & \text{otherwise}\end{cases}$$

**2.** Consider the function $f(x)=\begin{cases} x+2, & -2\le x<0 \\ 2, & 0\le x<2 \end{cases}$ $\quad f(x+4)=f(x)$

**a)** Sketch the graph of the given function for three periods.

**b)** Find the Fourier series for the given function.

**c)** Sketch the graph of the function to which the Fourier series converges for the same three periods.




b)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{2} + b_n \sin\frac{n\pi}{2}$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x) = \frac{1}{2}\int_{-2}^{0}(x+2)\,dx + \frac{1}{2}\int_{0}^{2} 2\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}+2x\right)\Big|_{-2}^{0} + \frac{1}{2}\,2(2) = -\frac{1}{2}\left(\frac{4}{2}-4\right) + 2 = 3$$

$$\boxed{a_0 = 3}$$

$$a_n = \frac{1}{2}\int_{-2}^{0}(x+2)\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} x\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{-2}^{0} 2\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} x\cos\frac{n\pi}{2}x\,dx + \int_{-2}^{2}\cos\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{n\pi}{2}\right)^2}\cos\frac{n\pi}{2}x\Big|_{-2}^{0} + \frac{x}{\frac{n\pi}{2}}\sin\frac{n\pi}{2}x\Big|_{-2}^{0}\right] + \frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big|_{-2}^{2}$$

$$a_n = \boxed{\frac{1}{2}\left[\frac{1}{\left(\frac{n\pi}{2}\right)^2}\left(1-\cos(n\pi)\right)}+0\right]+\frac{2}{n\pi}\,0 =$$

$$= \boxed{\frac{1}{2}\,\frac{4}{(n\pi)^2}\left(1-(-1)^n\right)} = \frac{2}{(n\pi)^2}\begin{cases}(2), & n=1,3,\dots\\ 0, & n=2,4,\dots\end{cases}$$

$$\Rightarrow \boxed{\begin{cases} a_{2n-1} = \dfrac{4}{[(2n-1)\pi]^2}, & n=1,2\dots \\ a_{2n} = 0, & n=1,2,3\dots \end{cases}}$$

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$$b_n = \frac{1}{2}\left[\int_{-2}^{0}(x+2)\sin\frac{n\pi}{2}x\,dx + \int_0^2 2\sin\frac{n\pi}{2}x\,dx\right]$$

$$= \frac{1}{2}\int_{-2}^{0} x\sin\left(\frac{n\pi}{2}x\right)dx + \frac{1}{2}\underbrace{\int_{-2}^{2} 2\sin\frac{n\pi}{2}x\,dx}_{=0}$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{n\pi}{2}\right)^2}\cancelto{0}{\sin\frac{n\pi}{2}x}\Bigg|_{-2}^{0} - \frac{x}{\frac{n\pi}{2}}\cos\frac{n\pi}{2}x\Bigg|_{-2}^{0}\right]$$

$$= -\frac{1}{2}\left[-\frac{2}{n\pi}(-2)\cos(-n\pi)\right] = -\frac{2}{n\pi}(-1)^n = (-1)^{n+1}\frac{2}{n\pi}$$

or $\boxed{b_n = (-1)^{n+1}\dfrac{2}{n\pi}}$

Or $\boxed{b_n = -\dfrac{2}{n\pi}\cos(-n\pi) = -\dfrac{2}{n\pi}\cos(n\pi)}$

**3.** Consider a rod of length $2$ for which $\alpha^2 = 2$. Let the rod be initially at the uniform temperature $10$. Suppose that at time $t = 0$, the end $x = 0$ is cooled to temperature $3$ and thereafter is maintained at this temperature, while the end $x = 2$ is insulated. Find the temperature distribution $u(x,t)$ in the rod at any time $t$.

The initial-boundary value problem modeling the above heat conduction problem in terms of the temperature distribution $u(x,t)$ at any moment of time is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t},\ 0 < x < 2,\ t > 0 \tag{1}$$

$$u(0,t) = 3,\ t > 0 \tag{2}$$

$$\frac{\partial}{\partial x}u(2,t) = 0,\ t > 0 \tag{3}$$

$$u(x,0) = 10,\ 0 \le x \le 2 \tag{4}$$

i) Find the steady state solution $u_s(s)$ of the given heat conduction equation (1) satisfying the boundary conditions (2)-(3).

ii) Express the temperature distribution $u(x,t)$ as the sum of the steady state temperature distribution $u_s(s)$ and the transient temperature distribution $U(x,t)$

$$u(x,t) = u_s(s) + U(x,t) \tag{5}$$

iii) Replace $u(x,t)$ in equations (1)-(4) using the equation (5), and obtain the initial-boundary value problem for the transient solution $U(x,t)$. That will be the basic initial-boundary value problem with homogeneous boundary conditions.

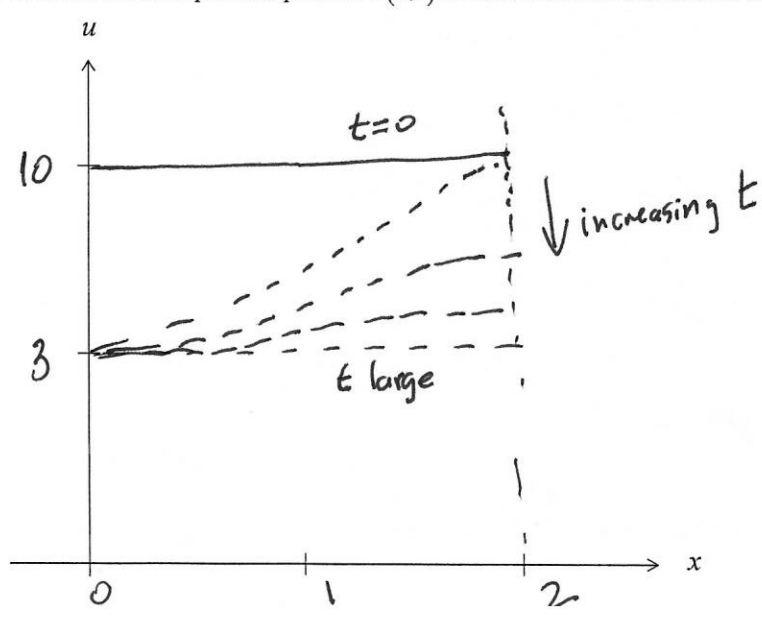
iv) Use the method of separation of variables to find the transient solution $U(x,t)$.

v) Write the solution $u(x,t)$ of the original problem (1)-(4) using equation (5).

Use the next pages for solution and write the final answer in the box below:

$u(x,t) =$

Sketch the graph of the initial temperature distribution and the steady state solution.
**Qualitatively** sketch several temperature profiles $u(x,t)$ for four different moments of time.



Quiz #4

#3) $$\begin{cases} u_t = 2u_{xx}, & 0<x<2,\ t>0 \\ u(0,t) = 3, \quad u_x(2,t)=0, & t>0 \\ u(x,0) = 10, & 0\le x\le 2. \end{cases}$$

<u>Steady-State soln:</u>

$v_{xx} = 0, \quad v(0)=3, \quad v'(2)=0$

$v(x) = c_1 x + c_2$

$3 = v(0) = c_2 \Rightarrow \boxed{c_2=3}$

$v'(x) = c_1 \Rightarrow v'(2) = c_1 = 0 \Rightarrow \boxed{c_1=0}$

$\Rightarrow \boxed{v(x)=3}$ steady-state soln.

Consider $w(x,t) = u(x,t) - v(x)$

$\Rightarrow w_t = u_t - \overset{0}{(v)_t} = 2u_{xx} = 2w_{xx} + 2\overset{=0}{v_{xx}} = 2w_{xx}$

or $\boxed{w_t = 2w_{xx}}$

Also, $w(0,t) = u(0,t) - v(0) = 3-3=0$

$w_x(2,t) = u_x(2,t) - v'(2) = 0-0=0$

and $w(x,0) = u(x,0) - v(x) = 10-3=7$

So, the transient IBVP:

$$\begin{cases} w_t = 2w_{xx}, & 0<x<2,\ t>0 \\ w(0,t)=0,\quad w_x(2,t)=0, & t>0 \\ w(x,0)=7, & 0<x<L \end{cases}$$

$$w(x,t) = X(x)T(t) \Rightarrow \begin{matrix} X''(x)+\lambda X(x)=0 \\ T'(t)+\lambda T(t)=0 \end{matrix}$$

Eigenvalue problem:

A) $\lambda>0$ $\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0,\quad X'(2)=0 \end{cases}$

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

$$0 = X(0) = C_1 \Rightarrow \boxed{C_1=0}$$

$$\boxed{X(x) = C_2 \sin\sqrt{\lambda}x}$$

$$X'(x) = \sqrt{\lambda}C_2 \cos\sqrt{\lambda}x \Rightarrow X'(2) = \sqrt{\lambda}C_2\cos\sqrt{\lambda}2 = 0$$

Then, $\overset{\neq 0}{\sqrt{\lambda}}C_2 = 0$ or $\cos\sqrt{\lambda}2 = 0$

If $C_2 = 0 \Rightarrow$ trivial soln. (not an eigenfn.)

Then $\cos\sqrt{\lambda}2 = 0 \Rightarrow \sqrt{\lambda}2 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

or $\boxed{\lambda_n = \left[\frac{(2n-1)\pi}{4}\right]^2}$

or $X_n(x) = C_n \sin\left(\frac{(2n-1)}{4}\pi x\right)$

and
$$T_n(t) = d_n\, e^{-\lambda_n t} = d_n\, e^{-\left(\frac{(2n-1)\pi}{4}\right)^2 t} \quad , n=1,2,\ldots$$

B) For $\lambda_0 = 0$

$$X''(x) = 0, \quad X(0) = 0, \quad X'(2) = 0$$

$$\Rightarrow \quad X(x) = C_1 x + C_2 \Rightarrow 0 = X(0) = C_2 \Rightarrow \boxed{C_2 = 0}$$

Also, $X'(x) = C_1$ and $X'(2) = 0 \Rightarrow \boxed{C_1 = 0}$

$\Rightarrow \lambda_0 = 0$ is not an eigenvalue.

C) For $\lambda < 0$, It can be proved that $X(x) = 0$ Not eigenfns.

General Soln:
$$\boxed{W(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\left(\frac{(2n-1)}{4}\pi\right)^2 t} \sin\left(\frac{(2n-1)}{4}\pi x\right)}$$

Using I.C.
$$7 = W(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)}{4}\pi x\right)$$

$$b_n = \frac{2}{2}\int_0^2 7 \sin\left(\frac{(2n-1)}{4}\pi x\right) dx$$

or $$b_n = 7\int_0^2 \sin\left(\frac{(2n-1)}{4}\pi x\right) dx = -7\left[\frac{4}{(2n-1)\pi} \cos\left(\frac{(2n-1)}{4}\pi x\right)\Big|_0^2\right]$$
$$= \frac{-28}{(2n-1)\pi}\left(\cos\frac{(2n-1)\pi}{2} - \cos(0)\right) = \frac{28}{(2n-1)\pi}$$

Therefore,

$$\boxed{w(x,t) = \sum_{n=1}^{\infty} \frac{28}{(2n-1)\pi} e^{-\left(\frac{(2n-1)}{4}\pi\right)^2 t} \sin\left(\frac{(2n-1)}{4}\pi x\right)}$$

and

$$u(x,t) = w(x,t) + v(x)$$

or

$$\boxed{u(x,t) = 3 + \sum_{n=1}^{\infty} \frac{28}{(2n-1)\pi} e^{-\left(\frac{(2n-1)\pi}{4}\right)^2 t} \sin\left(\frac{(2n-1)}{4}\pi x\right)}$$