7.7.9 Circularly Symmetric Case.

$$u(r,\theta,t) = u(r,t)$$

$$\frac{\partial}{\partial\theta} = 0. \Rightarrow \nabla^2_{r,\theta} u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}}$$

Then IBVP reduces to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{c^2}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) & (1.1) \\ u(a,t) = 0 & (1.2) \\ u(r,0) = \alpha(r), \quad \dfrac{\partial u}{\partial t}(r,0) = \beta(r) & (1.3) \end{cases}$$

Separating variables:

$$\boxed{u(r,t) = \phi(r)\, h(t)} \qquad (1.4)$$

$$\frac{d^2 h(t)}{dt^2}\phi(r) = \frac{c^2}{r} h(t) \frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$$

dividing by $c^2 h(t)\phi(r)$.

$$\frac{1}{h(t)}\frac{1}{c^2}\frac{d^2 h}{dt^2} = \frac{1}{r\phi(r)}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda$$

Time equation:

$$\boxed{\frac{d^2 h}{dt^2} + \lambda c^2 h = 0} \qquad (1.5)$$

Eigenvalue Problem:

$$\begin{cases} \dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right) + \lambda r\phi = 0 & (2.1) \\ \phi(a) = 0, \quad |\phi(0)| < \infty & (2.2) \end{cases}$$

A Rayleigh Quotient can be obtained for EVP (2.1)-(2.2) by multiplicating by $\phi(r)$ and $\int_0^a dr$. By doing this, we obtain

$$\lambda = \frac{\int_0^a r\left(\frac{d\phi}{dr}\right)^2 dr}{\int_0^a \phi^2(r)\, r\,dr} \geq 0 \qquad (2.3)$$

Prove this!

(See exercise 5.5.1 (f))

and from (2.2), $\underline{\underline{\lambda > 0}}$ only possibility.

Following a similar procedure as the one employed for the more general case, ① multiply (2.1) by r: $r^2\phi'' + r\phi' + \lambda r^2\phi = 0$ ② we introduce the new variable

$$\boxed{z = \sqrt{\lambda}\, r}$$

and equation (2.1) is transformed into Bessel's differential equation of order zero.

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \lambda r^2\phi = 0 \xleftrightarrow{z=\sqrt{\lambda}r} z^2\frac{d^2\hat{\phi}}{dz^2} + z\frac{d\hat{\phi}}{dz} + z^2\hat{\phi} = 0.$$

where $\hat{\phi}(z) = \hat{\phi}(\sqrt{\lambda}r) = \phi(r)$

Proof of inequality (2.3).

Multiplying (2.1) by $\phi(r)$ and integrating $\int_0^a dr$, we obtain

$$\int_0^a \phi \frac{d}{dr}\left(r \frac{d\phi}{dr}\right) dr + \lambda \int_0^a \phi^2(r)\, r\, dr = 0$$

$$\Rightarrow \qquad \lambda = \frac{-\int_0^a \phi \frac{d}{dr}\left(r \frac{d\phi}{dr}\right) dr}{\int_0^a \phi^2(r)\, r\, dr} \qquad (2.4)$$

Now, $\frac{d}{dr}\left[\phi\, r \frac{d\phi}{dr}\right] = r\left(\frac{d\phi}{dr}\right)^2 + \phi \frac{d}{dr}\left(r \frac{d\phi}{dr}\right)$

$$\Rightarrow \quad \phi \frac{d}{dr}\left(r \frac{d\phi}{dr}\right) = \frac{d}{dr}\left[\phi\, r \frac{d\phi}{dr}\right] - r\left(\frac{d\phi}{dr}\right)^2.$$

Subst. into (2.4)

$$\lambda = \frac{-\int_0^a \frac{d}{dr}\left[\phi r \frac{d\phi}{dr}\right] + \int_0^a \left(\frac{d\phi}{dr}\right)^2 r\, dr}{\int_0^a \phi^2(r)\, r\, dr}$$

$$\Rightarrow \quad \lambda = \frac{-\phi(r)\, r \frac{d\phi}{dr}(r)\Big|_0^a + \int_0^a \left(\frac{d\phi}{dr}\right)^2 r\, dr}{\int_0^a \phi^2(r)\, r\, dr}$$

Now, $|\phi(0)| < \infty$ and if we assume

$$\lim_{r\to 0} r\frac{d\phi}{dr}(r) = 0$$

then

$$\phi(r)\, r\, \frac{d\phi}{dr}(r)\Big|_0^a = \cancel{\phi(a)}^{\,0}\, a\, \frac{d\phi}{dr}(a) - \phi(0)\, 0\, \cancel{\frac{d\phi}{dr}(0)}^{\,\to 0}$$

$$= 0$$

Thus,

$$\lambda = \frac{\int_0^a \left(\frac{d\phi}{dr}\right)^2 r\, dr}{\int_0^a \phi^2(r)\, r\, dr} \geq 0$$

$\lambda = 0 \iff \frac{d\phi}{dr}(r) \equiv 0 \Rightarrow \phi(r) \equiv \text{const.}$

Using B.C.' at $r = a$ $\quad \phi(a) = 0$

$\Rightarrow \phi(r) \equiv 0 \Rightarrow \lambda = 0$ is not an eigenvalue.

Therefore, $\underline{\underline{\lambda > 0}}$.

The general solution of Bessel's ODE of order zero:

$$z^2 \frac{d^2\hat{\phi}}{d^2} + z\frac{d\hat{\phi}}{d} + z^2\hat{\phi} = 0$$

is given by

$$\boxed{\hat{\phi}(z) = C_1 J_0(z) + C_2 Y_0(z)}$$

and returning the change of variables:

$$\boxed{\phi(r) = C_1 J_0(\sqrt{\lambda}r) + C_2 Y_0(\sqrt{\lambda}r)}$$

Since $|\phi(0)| < \infty \Rightarrow C_2 = 0.$

and $\boxed{\phi(r) = C_1 J_0(\sqrt{\lambda}r)}$

Using the B.C. at $r = a$.

$$J_0(\sqrt{\lambda}a) = 0 \Rightarrow \sqrt{\lambda}a = z_{0n}, \quad n = 1, 2, \ldots$$

zeros of Bessel fn. of order zero.

$$\Rightarrow \left\{ \lambda_n = \left(\frac{z_{0n}}{a}\right)^2, \quad n = 1, 2, \ldots \right\} \text{ eigenvalues.}$$

and the eigenfunctions are

$$\left\{ J_0\left(\frac{z_{0n}}{a} r\right) \right\}_{n=1}^{\infty}.$$

The time equation (1.5) has the general solution: $(\lambda > 0)$

$$\boxed{h(t) = A_n \sin(\sqrt{\lambda_n}\, ct) + B_n \cos(\sqrt{\lambda_n}\, ct).} \quad (4.1)$$

Then, Product solns. are given by

$$\left[A_n \sin\left(\frac{z_{0n}}{a} ct\right) + B_n \cos\left(\frac{z_{0n}}{a} ct\right)\right] J_0\left(\frac{z_{0n}}{a} r\right)$$

Using the pple of Superposition the final Series soln. is given by

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{z_{0n}}{a} ct\right) + B_n \cos\left(\frac{z_{0n}}{a} ct\right)\right] J_0\left(\frac{z_{0n}}{a} r\right)}$$

An and Bn are determined using the orthogonality (prove it!) of the Bessel's function of order zero.

In fact,

$$\alpha(r) = u(r,0) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{z_{0n}}{a} r\right)$$

multiplying by $J_0\left(\frac{z_{0m}}{a} r\right) r$ and $\int_0^a dr$ and Solving for Bn.

$$B_n = \frac{\int_0^a \alpha(r) J_0\left(\frac{z_{0n}}{a} r\right) r\, dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\, dr}, \quad n = 1, 2, \ldots$$

Similarly, An's are obtained.. from the I.C. $u_t(r,0) = \beta(r)$.

$$U_t(r,t) = \sum_{n=1}^{\infty} \frac{c\, z_{0n}}{a}\left[A_n \cos\left(\frac{z_{0n}}{a}ct\right) - B_n \sin\left(\frac{z_{0n}}{a}ct\right)\right] J_0\left(\frac{z_{0n}}{a}r\right)$$

Using I.C.

$$\beta(r) = U_t(r,0) = \sum_{n=1}^{\infty} \frac{c\, z_{0n}}{a} A_n J_0\left(\frac{z_{0n}}{a}r\right)$$

$$\Rightarrow \quad A_n = \frac{a}{c\, z_{0n}} \frac{\int_0^a \beta(r) J_0\left(\frac{z_{0n}}{a}r\right) r\, dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a}r\right) r\, dr}, \quad n = 1, 2, \ldots$$

Consider the S-L EVP (Not regular)

$$\begin{cases} \dfrac{d}{dr}\left(r\dfrac{df}{dr}\right)+\left(\lambda_m r-\dfrac{m^2}{r}\right)f=0, & 0<r<a \quad (1) \\ & m \text{ fixed} \\ |f(0)|<\infty, \quad f(a)=0 & (2) \end{cases}$$

obtained from the Separ. of Variables for the Vibrations of a Circular membrane.

We showed that there are an <u>infinite set of eigenvalues</u> given by

$$\lambda_{mn}=\left(\frac{z_{mn}}{a}\right)^2, \quad m \text{ fixed}, \; n=1,2,\ldots \qquad (3)$$

where z_{mn} are the roots (all real) of $J_m(z)$ (Bessel function of order m).

We also obtained the <u>corresponding eigenfunctions</u>:

$$J_m\left(\frac{z_{mn}}{a}r\right), \quad m \text{ fixed}, \; n=1,2,\ldots \qquad (4)$$

<u>Theorem</u>:- Eigenfunctions of eigenvalue problem (1)-(2) corresponding to different eigenvalues are orthogonal.

Proof:- Consider two different eigenvalues:

λ_{mn} and λ_{ml} of EVP (1)-(2)

with corresponding eigenfunctions

ϕ_{mn} and ϕ_{ml}

What we want to prove is that

$$\int_0^a J_m\left(\frac{z_{mn}}{a}r\right) J_m\left(\frac{z_{ml}}{a}r\right) \sigma(r)\, dr \overset{?}{=} 0$$

Ask for ideas!

Start with Green's formula for S-L differential operator

Call $L \equiv \frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{m^2}{r}$

Green's form: u, v any two suff. smooth functions

$$\int_0^a (uLv - vLu)\, dr = r\left(u\frac{dv}{dr} - v\frac{du}{dr}\right)\Bigg|_0^a$$

Now, define $u = \phi_{mn}$ and $v = \phi_{ml}$.

$$\boxed{\begin{aligned} L\phi_{mn} &= -\lambda_{mn} r \phi_{mn} \\ L\phi_{ml} &= -\lambda_{ml} r \phi_{ml} \end{aligned}}$$

Then,

$$\int_0^a \left[\phi_{mn} L\phi_{ml} - \phi_{ml} L \phi_{mn}\right] dx = r\left(\phi_{mn}\frac{d\phi_{ml}}{dr} - \phi_{ml}\frac{d\phi_{mn}}{dr}\right)\Bigg|_0^a$$

$$= a\left[\overset{0}{\phi_{mn}(a)}\frac{d\phi_{ml}}{dr}(a) - \overset{0}{\phi_{ml}(a)}\frac{d\phi_{mn}}{dr}(a)\right] - 0\left[\overset{<\infty}{\phi_{mn}(0)}\phi_{ml}'(0) - \phi_{ml}(0)\overset{<\infty}{\phi_{mn}'(0)}\right]$$

$= 0$

Thus,

$$\int_0^a \left[\phi_{mn} L \phi_{ml} - \phi_{ml} L \phi_{mn}\right] dr = 0$$

$$\Leftrightarrow \quad \int_0^a \left[\phi_{mn}\left(-\lambda_{ml} r \phi_{ml}\right) - \phi_{ml}\left(-\lambda_{mn} r \phi_{mn}\right)\right] dr = 0$$

$$\Leftrightarrow \quad \left(\lambda_{mn} \overset{\neq 0}{-} \lambda_{ml}\right) \int_0^a \phi_{mn} \phi_{ml} r \, dr = 0$$

$$\Rightarrow \quad \int_0^a \phi_{mn} \phi_{ml} r \, dr = \int_0^a J_m\left(\frac{z_{mn}}{a} r\right) J_m\left(\frac{z_{ml}}{a}\right) r \, dr = 0 \checkmark$$

Green's formula for differential operator:

$$L \equiv \frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{m^2}{r}$$

$$uLv - vLu = u\frac{d}{dr}\left(r\frac{dv}{dr}\right) - \cancel{\frac{m^2}{r}uv} - u\frac{d}{dr}\left(r\frac{du}{dr}\right) + \cancel{\frac{m^2}{r}uv}$$

$$\frac{d}{dr}\left(ur\frac{dv}{dr}\right) = r\frac{du}{dr}\frac{dv}{dr} + u\frac{d}{dr}\left(r\frac{dv}{dr}\right)$$

$$\Rightarrow \quad u\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \frac{d}{dr}\left(ur\frac{dv}{dr}\right) - r\frac{du}{dr}\frac{dv}{dr}$$

Similarly, $v\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{d}{dr}\left(vr\frac{du}{dr}\right) - r\frac{du}{dr}\frac{dv}{dr}$

Thus,

$$u\frac{d}{dr}\left(r\frac{dv}{dr}\right) - v\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{d}{dr}\left[ur\frac{dv}{dr} - vr\frac{du}{dr}\right]$$

$\int_0^a dr$

$$\Rightarrow \int_0^a \left[u\frac{d}{dr}\left(r\frac{dv}{dr}\right) - v\frac{d}{dr}\left(r\frac{du}{dr}\right)\right] dr = r\left(u\frac{dv}{dr} - v\frac{du}{dr}\right)\Big|_0^a$$

Use this formula to prove orthogonality of eigenfns: $\phi_{nm}(r)$ of eigenvalue problem (1.2) and (1.3).

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~~equivalently~~

$$\cancel{0 = \int_0^a \left[\phi_{mn}(-\lambda_{mn} r \phi_{m\ell}) - \phi_{m\ell}(-\lambda_{m\ell} r \phi_{mn})\right] dr =}$$

$$\cancel{= (\lambda_{m\ell} - \lambda_{mn}) \int_0^a \phi_{mn}\phi_{m\ell}\, r\, dr}$$

$$\cancel{\Rightarrow \int_0^a \phi_{mn}\phi_{m\ell}\, r\, dr = 0} \checkmark$$

~~For Neumann problem, the rhs of (2.2) cancels out at $r=a$ since $\frac{d\phi}{dr}(a) = 0$. At $r=0$, it's identical to the Dirichlet case. The rest of the proof is identical to the Dirichlet problem's proof.~~

Positiveness of Eigenvalues of ∇^2 operator with Dirichlet BC's.

$$\begin{cases} \nabla^2\phi + \lambda\phi = 0, & \text{on } \Omega \\ \phi = 0, & \text{on } \partial\Omega \end{cases}$$

$\nabla\cdot(\phi\nabla\phi) = \phi\nabla^2\phi + \|\nabla\phi\|^2$

$$\phi\nabla^2\phi + \lambda\phi^2 = 0 \Rightarrow \int_\Omega \phi\nabla^2\phi\, dv + \lambda\int_\Omega \phi^2 dv = 0 \qquad (3.1)$$

Now, $\int_\Omega \phi\nabla^2\phi\, dv = \int \overset{\nearrow 0}{\phi\nabla\phi}\, ds - \int_\Omega \|\nabla\phi\|^2 dv = -\int_\Omega \|\nabla\phi\|^2 dv$

From (3.1)

$$\Longrightarrow \boxed{\lambda = \frac{\int_\Omega \|\nabla\phi\|^2 dv}{\int_\Omega \phi^2 dv} \geq 0}$$

If $\lambda = 0 \Rightarrow \nabla\phi \equiv 0 \Rightarrow \phi \equiv \text{const}$
but $\phi(\vec{x}) = 0$ at $\partial\Omega$
$\Rightarrow \phi(\vec{x}) \equiv 0$ and $\lambda = 0$ is not an eigenvalue