7.7. Vibrating Circular Membrane and Bessel Functions.

Consider the IBVP:

$$\begin{cases} u_{tt} = c^2 \nabla^2_{r,\theta} u, & 0<r<a, \quad t>0. \quad (1.1) \\ \text{B.C.}: u(a,\theta,t) = 0 & (1.2) \\ \text{I.C's} \begin{cases} u(r,\theta,0) = \alpha(r,\theta) & (1.3) \\ u_t(r,\theta,0) = \beta(r,\theta) & (1.4) \end{cases} \end{cases}$$

Separation of Variables:

$$u(r,\theta,t) = \phi(r,\theta)\, h(t)$$

$$h''(t)\,\phi(r,\theta) = c^2 \nabla^2_{r,\theta}\phi \; h(t)$$

Dividing by

$$\frac{h''(t)}{c^2 h(t)} = \frac{\nabla^2_{r,\theta}\phi}{\phi} = -\lambda$$

ODE for $h(t)$:

$$\boxed{h''(t) + \lambda c^2 h(t) = 0},$$

B.C. $u(a,\theta,t) = 0 \Leftrightarrow$
$h(t)\,\phi(a,\theta) = 0 \Rightarrow$
$\boxed{\phi(a,\theta) = 0}$.

Eigenvalue problem for $\phi(r,\theta)$

$$\begin{cases} \nabla^2_{r,\theta}\phi + \lambda\phi = 0 & (1.5) \\ \phi(a,\theta) = 0 & (1.6) \end{cases}$$

The solution for $h(t)$ is given by

$$\boxed{h(t) = A \cos c\sqrt{\lambda}t + B \sin c\sqrt{\lambda}t} \tag{2.1}$$

To solve EVP (1.5)-(1.6), a new separation of variables is performed

$$\boxed{\phi(r,\theta) = f(r)\, g(\theta)} \tag{2.2}$$

$$0 = \nabla^2_{r,\theta}\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$= \frac{g(\theta)}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{f(r)}{r^2}\frac{d^2 g}{d\theta^2} + \lambda f(r)\, g(\theta)$$

Then multiplying by $\dfrac{r^2}{f(r)g(\theta)}$

$$\boxed{-\frac{1}{g(\theta)}\frac{d^2 g}{d\theta^2} = \frac{r}{f(r)}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \lambda r^2 = \mu} \tag{2.3}$$

From (2.3) and additional BC's two new EVP results:

$$\begin{cases} \dfrac{d^2 g}{d\theta^2} + \mu\, g(\theta) = 0 & (2.4) \\ g(-\pi) = g(\pi) & (2.5) \\ \dfrac{dg}{d\theta}(-\pi) = \dfrac{dg}{d\theta}(\pi) & (2.6) \end{cases}$$

$$\begin{cases} r\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0 & (2.7) \\ f(a) = 0 & (2.8) \\ |f(0)| < \infty & (2.9) \end{cases}$$

The EVP. (2.4)-(2.6) is well-known and the eigenvalues and eigenvectors are given by

$$\boxed{\mu_m = m^2}, \quad m=0,1,2,\dots \tag{3.1}$$

$$g_m(\theta) = \begin{cases} \sin(m\theta) \\ \cos(m\theta) \end{cases} \quad m=0,1,2,\dots \tag{3.2}$$

For each $m=1,2,\dots$, the EVP (2.7)-(2.9) need to be solved

$$\begin{cases} r\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + (\lambda r^2 - m^2) f = 0 & (3.3) \\ f(a)=0, \quad |f(0)|<\infty. & (3.4) \end{cases}$$

EVP (3.3)-(3.4) is not a regular Sturm-Liouville EVP.

In fact, dividing by r (3.3) is reduced to

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0$$

Therefore, $p(r)\equiv r$, $\sigma(r)\equiv r$, and $q(r)\equiv \frac{-m^2}{r}$

$p(0)=0$, $\sigma(0)=0$ violating that $p(r)>0$, and $\sigma(r)>0$

where $0\le r\le a$. Also $q(r)$ is discontinuous at $\underline{\underline{r=0}}$.

However, it is possible to prove that EVP (3.3)-(3.4) Satisfies all the properties of regular Sturm-Liouville problems.

7.7.4 Bessel's Differential Equation.

Equation (3.3) Can be transformed into

$$(4.1) \quad \boxed{z^2 \frac{d^2\hat{f}}{dz^2} + z\frac{d\hat{f}}{dz} + (z^2 - m^2)\hat{f} = 0}$$

Bessel's differential equation of order m.

In fact, by defining : $\boxed{z = \sqrt{\lambda}\, r} \Rightarrow r = \frac{z}{\sqrt{\lambda}}$

$$f(r) = f\left(\frac{z}{\sqrt{\lambda}}\right) = \hat{f}(z), \quad \text{or} \quad \hat{f}(z) = \hat{f}(\sqrt{\lambda}\, r) = f(r)$$

$$\Rightarrow \frac{df}{dr} = \frac{d\hat{f}}{dz}\frac{dz}{dr} = \sqrt{\lambda}\frac{d\hat{f}}{dz} \Rightarrow \frac{d^2 f}{dr^2} = \sqrt{\lambda} \cdot \frac{d^2\hat{f}}{dz^2}\frac{dz}{dr} =$$

$$= \lambda \frac{d^2\hat{f}}{dz^2}.$$

First, we express (3.3) as

$$r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - m^2) f = 0.$$

Subst. of the new Variables into this equation leads to

$$\frac{z^2}{\cancel{\lambda}}\cancel{\lambda}\frac{d^2\hat{f}}{dz^2} + \frac{z}{\cancel{\sqrt{\lambda}}}\cancel{\sqrt{\lambda}}\frac{d\hat{f}}{dz} + (z^2 - m^2)\hat{f} = 0$$

or

$$\boxed{z^2 \frac{d^2 \hat{f}}{dz^2} + z \frac{d\hat{f}}{dz} + (z^2 - m^2)\hat{f} = 0} \qquad (5.1)$$

$z = 0$ is a singular point. We expect $\hat{f}(z)$ to behave bad about $z=0$.

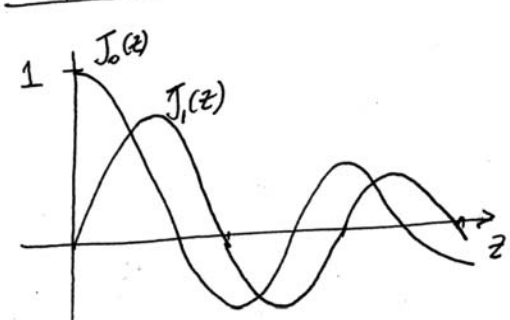
Equation (5.1) has two linearly independent Solutions:

$J_m(z)$: Bessel function of the first kind of Order m. well-behaved at $z=0$.

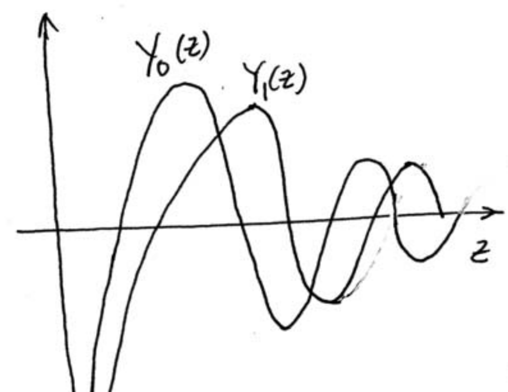
$Y_m(z)$: " " " the second kind of order m.

$Y_m(z) \xrightarrow[z \to 0]{} \pm\infty$ (bad behaved)

Graphs:



Decaying oscillations bounded at $z=0$. Infinitely many roots



Decaying oscillations unbounded at $z=0$. Infinitely many roots.

Thus, the general Solution of (4.1) Can be written as

$$\boxed{\hat{f}(z) = C_1 J_m(z) + C_2 Y_m(z).} \qquad (6.1)$$

Returning to our EVP,

$$\boxed{f(r) = C_1 J_m(\sqrt{\lambda} r) + C_2 Y_m(\sqrt{\lambda} r)} \qquad (6.2)$$

Now, $|f(0)| < \infty$

$\Rightarrow \quad C_2 = 0$ and $\boxed{f(r) = C_1 J_m(\sqrt{\lambda} r).}$

and using the other B.C. $\quad f(a) = 0$

$$\Rightarrow \qquad f(a) = C_1 J_m(\sqrt{\lambda}\, a) = 0$$

Therefore, $\quad \sqrt{\lambda}\, a =$ "root of $J_m(z)$"

Since $J_m(z)$ has infinitely many roots : $\{Z_{mn}\}_{n=1}^{\infty}$ m fixed.

$$\Rightarrow \qquad \sqrt{\lambda}\, a = Z_{mn}, \qquad n = 1, 2, \ldots$$

$$\Rightarrow \qquad \boxed{\lambda_{mn} = \frac{Z_{mn}^2}{a^2}} \qquad \begin{array}{l} m \text{ fixed } (m = 0, 1, 2 \ldots) \text{ Eigenvalues} \\ n = 1, 2, 3, \ldots \end{array} \qquad (6.3)$$

With corresponding eigenfunctions:

$$\boxed{J_m\left(\sqrt{\lambda_{mn}}\, r\right) = J_m\left(\frac{Z_{mn}}{a} r\right)} \qquad \begin{array}{l} m \text{ fixed} \\ n = 1, 2, \ldots \end{array} \qquad (6.4)$$

Comparison with Rectangular Membrane EVP:

1-D Cartesian

$$\begin{cases} \dfrac{d^2 g}{dy^2} + (\lambda - \mu_n) g = 0 & \mu_n = \left(\frac{n\pi}{L}\right)^2 \\ g(0) = 0, \quad g(H) = 0 & \end{cases}$$

n fixed

Eigenvalues:

$$\lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2$$

or $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

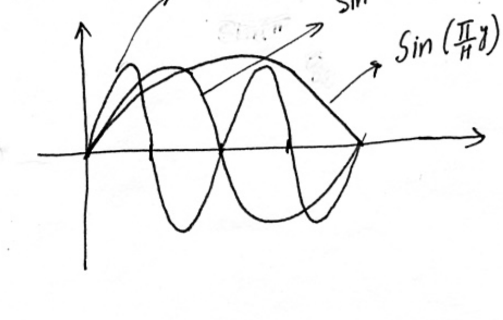
n fixed, $m = 1, 2, 3 \ldots$

Eigenfunctions: n fixed, $m = 1, 2, \ldots$

$$g_{nm}(y) = \sin\left(\frac{m\pi}{H} y\right)$$

where $\left\{ \begin{matrix} z_m = \\ m\pi \end{matrix} \right\}_{m=1}^{\infty}$ are the roots of $\sin(z)$

Graphs:



1-D radial variable.

$$\begin{cases} \dfrac{d}{dr}\left(r \dfrac{df}{dr}\right) + \left(\lambda_m r - \dfrac{m^2}{r}\right) f = 0 \\ f(a) = 0, \quad |f(0)| < \infty \end{cases}$$

m fixed

EIGENVALUES:

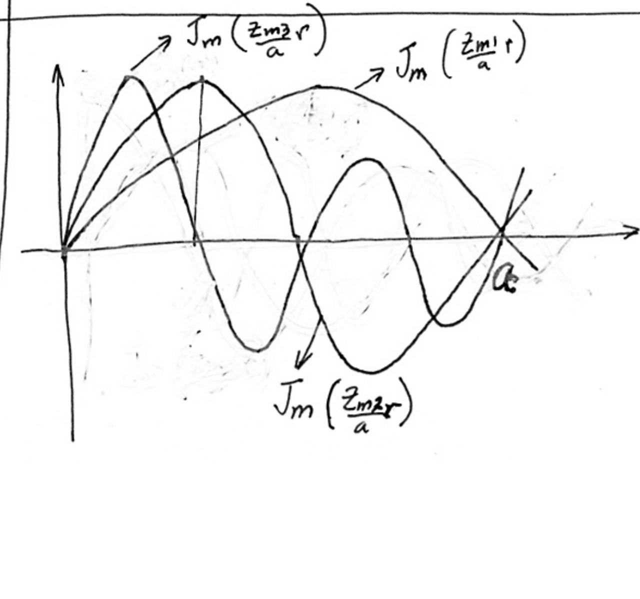
$$(\lambda_m)_n = \frac{z_{mn}^2}{a^2}, \quad \begin{matrix} m \text{ fixed} \\ n = 1, 2, \ldots \end{matrix}$$

EIGENFUNCTIONS: m fixed, $n = 1, 2, \ldots$

$$f_{mn}(r) = J_m\left(\sqrt{\lambda_{mn}}\, r\right) =$$

$$= J_m\left(\frac{z_{mn}}{a} r\right)$$

Where $\{z_{mn}\}_{n=1}^{\infty}$ are the roots of $J_m(z)$.



The Bessel's eigenfunctions happen to be orthogonal for each fixed m. In fact,

weight

$$\boxed{\int_0^a J_m\left(\sqrt{\lambda_{mp}}\, r\right) J_m\left(\sqrt{\lambda_{mq}}\, r\right) r \, dr = 0,} \quad \text{If } p \neq q \qquad (8.1)$$

Also, for m fixed they form a complete set for piecewise smooth functions of r.

Any p.w.s. function

$$\boxed{h(r) \sim \sum_{n=1}^{\infty} a_n J_m\left(\sqrt{\lambda_{mn}}\, r\right) \quad m \text{ fixed}} \qquad (8.2)$$

Fourier - Bessel Series.

Using orthogonality the coefficients "a_n" can be determined.

$$\boxed{a_n = \frac{\int_0^a h(r) J_m\left(\sqrt{\lambda_{mn}}\, r\right) r \, dr}{\int_0^a J_m^2\left(\sqrt{\lambda_{mn}}\, r\right) r \, dr}} \qquad (8.3).$$

Going back to our original IBVP:

$$\begin{cases} u_{tt} = c^2 \nabla^2_{r,\theta} u = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right] \\ u(a,\theta,t) = 0, \quad u(r,\theta,0) = \alpha(r,\theta), \quad u_t(r,\theta,0) = \beta(r,\theta) \end{cases}$$

After the separation of variables, we have obtained the <u>product solutions</u>:

$$J_m\left(\sqrt{\lambda_{mn}}\, r\right) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \begin{Bmatrix} \cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t \end{Bmatrix}.$$

Using the principle of superposition, the solution can be represented by

$$\boxed{\begin{aligned} u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Big[A_{mn}\, J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) \;+ \\ B_{mn}\, J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \Big] \cos\left(c\sqrt{\lambda_{mn}}\, t\right) + \\ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Big[C_{mn}\, J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) + D_{nm}\, J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \Big] \sin\left(c\sqrt{\lambda_{mn}}\, t\right) \end{aligned}}$$

Four families of unknown coefficients to be determined from the I.C's.