Outline for a final class of Circular membrane. (wave equ.)

Good Summary

1) Set up. IBVP.

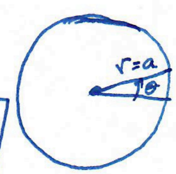
2) Sep. of Vabs: $u(r,\theta,t) = h(t)\,\phi(r,\theta)$

t-equ. : $h''(t) + \lambda c^2 h(t) = 0$

Solns: $\boxed{h(t) = A\cos c\sqrt{\lambda}t + B\sin c\sqrt{\lambda}t.}$ for $\lambda \geq 0$

Space equ: Eigenvalue Problem:

$$\begin{cases} \nabla^2_{r,\theta}\phi + \lambda\phi = 0 \iff \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0 \\ \phi(a,\theta) = 0 \end{cases}$$



3) Further Separation of Space Variables: $\boxed{\phi(r,\theta) = f(r)\,g(\theta)}$

Two Eigenvalue problems:

$$\begin{cases} \dfrac{d^2g}{d\theta^2} + \mu g(\theta) = 0 \\ g(-\pi) = g(\pi) \\ \dfrac{dg}{d\theta}(-\pi) = \dfrac{dg}{d\theta}(\pi) \end{cases}$$

Not regular S-L EVP.
B.C's.

$$\begin{cases} r\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + (\lambda r^2 - \mu)f = 0 \\ f(a) = 0 \\ |f(0)| < \infty \end{cases}$$

Not regular SL-EVP, but still Good properties. of regular problems.

4) Eigenvalues and Eigenfunctions for $g(\theta)$:

$$\mu_m = m^2, \quad m = 0,1,2,\ldots$$

$$g_m(\theta) = \left\{ \begin{matrix} \sin(m\theta) \\ \cos(m\theta) \end{matrix} \right\}_{m=0}^{\infty}$$

5) Eigenvalues and Eigenfunctions for $f(r)$:

a) Transformation to Bessel's differential equation:

$$z = \sqrt{\lambda}\, r \longrightarrow \qquad r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - m^2) f = 0$$

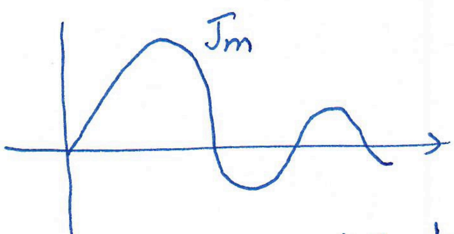
$$\downarrow$$

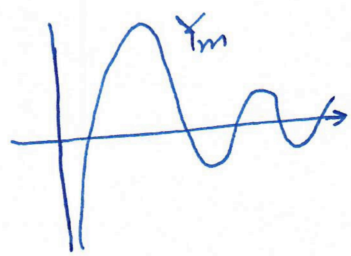
$$z^2 \frac{d^2 \hat{f}}{dz^2} + z\frac{d\hat{f}}{dz} + (z^2 - m^2)\hat{f} = 0$$

General Solution: $\hat{f}(z) = C_1 J_m(z) + C_2 Y_m(z)$.

$$z = \sqrt{\lambda}\, r \longrightarrow \quad \downarrow \; = \hat{f}(z(r))$$

$$f(r) = C_1 J_m(\sqrt{\lambda}\, r) + C_2 Y_m(\sqrt{\lambda}\, r).$$





Since $|f(0)| < \infty$

$$\Rightarrow \boxed{f(r) = C_1 J_m(\sqrt{\lambda r})}$$

B.c: $\quad u(a,\theta,t)=0 \Rightarrow f(a)=0.$

$\therefore$ For each m,

$$0 = f(a) = c_1 J_m(\sqrt{\lambda}\, a) = 0$$

$$\Rightarrow \sqrt{\lambda}\, a = \text{"root of } J_m(z)\text{"}$$

$$\Rightarrow \sqrt{\lambda}\, a = z_{mn}, \quad \begin{array}{l} m \text{ fixed} \\ n=1,2,\ldots \end{array} \Rightarrow \boxed{\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2.}$$

and Corresponding eigenfunctions are

$$\boxed{J_m(\sqrt{\lambda_{mn}}\, r) = J_m\left(\frac{z_{mn}}{a} r\right)}. \quad \begin{array}{l} m \text{ fixed} \\ n=1,2,\ldots \end{array}$$

DISCUSS COMPLETENESS AND ORTHOGONALITY (page 8 of notes).

6) Eigenvalues and Eigenfns. for 2-D EVP:

$$\boxed{\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2}, \quad \boxed{\phi_{mn} = J_m\left(\frac{z_{mn}}{a} r\right)\begin{Bmatrix} \sin(m\theta) \\ \cos(m\theta) \end{Bmatrix} \begin{array}{l} m=0,1,2\ldots \\ n=1,2,\ldots \end{array}}$$

7) Product Solns:

$$J_m(\sqrt{\lambda_{mn}}\, r)\begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}\begin{Bmatrix} \cos(c\sqrt{\lambda_{mn}}\, t) \\ \sin(c\sqrt{\lambda_{mn}}\, t) \end{Bmatrix}$$

8) Final Series Solution:

$$u(r,\theta,t) =$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos(m\theta) + B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\right]\cos(c\sqrt{\lambda_{mn}}\, t) +$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[C_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos(m\theta) + D_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\right]\sin(c\sqrt{\lambda_{mn}}\, t)$$

Special IBVP.

$$\begin{cases} u_{tt} = c^2 \nabla^2_{r,\theta} u, \\ u(a, \theta, t) = 0 \quad (a=1) \\ u(r,\theta,0) = J_3(\sqrt{\lambda_{32}}\, r)\cos 3\theta = J_3(z_{32} r)\cos 3\theta \\ u_t(r,\theta,0) = 0 \end{cases}$$

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos(m\theta) + B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\right]\cos(c\sqrt{\lambda_{mn}}\, t)$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[C_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos(m\theta) + D_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\right]\sin(c\sqrt{\lambda_{mn}}\, t)$$

Applying I.C's $0 = u_t(r,\theta,0) = \sum_m \sum_n \sqrt{\lambda_{nm}}\, c\left[C_{mn}\cdots + D_{nm}\cdots\right] \Rightarrow \begin{matrix} C_{mn}=0 \\ D_{mn}=0 \end{matrix}$

$$J_3(z_{32} r)\cos 3\theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn} \overset{\cos}{\cdots} + B_{mn} \overset{\sin}{\cdots}\right]\cdot 1$$

$\Rightarrow$ $A_{32} = 1$, $A_{mn} = 0$, otherwise

$B_{mn} = 0$.

$$\Rightarrow \boxed{u(r,\theta,t) = J_3(\sqrt{\lambda_{32}}\, r)\cos 3\theta \cos(c\sqrt{\lambda_{32}}\, t).}$$