Complex Form of Fourier Series.

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\,dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\,dx$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\Rightarrow \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

If $\quad \theta_n \equiv \frac{n\pi}{L}$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{\boxed{e^{i\theta_n x}} + e^{-i\theta_n x}}{2} \right) + b_n \left(\frac{\boxed{e^{i\theta_n x}} - e^{-i\theta_n x}}{2i} \right) \right]$$

or

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} \left[(a_n - ib_n) e^{i\theta_n x} + (a_n + ib_n) e^{-i\theta_n x} \right]$$

Then,

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - ib_n) e^{i\theta_n x} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-i\theta_n x}$$

Calling $m = -n$

$$f(x) \sim a_0 + \frac{1}{2} \sum_{m=-1}^{-\infty} (a_{-m} - ib_{-m}) e^{i\theta_{-m} x} + \frac{1}{2} (\quad)$$

Now, $a_{-m} \overset{\text{def of } a_m}{=} a_m$, $\quad b_{-m} \overset{\text{def } b_m}{=} -b_m$, $\quad \theta_{-m} = -\theta_m$

Then,

$$f(x) \sim a_0 + \frac{1}{2} \sum_{m=-\infty}^{-1} (a_m + ib_m) e^{-i\theta_m x} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-i\theta_n x}$$

renaming $m = n$ and using $b_0 = 0$.

$$f(x) \sim a_0 + \overset{=0}{b_0} + \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (a_n + ib_n) e^{-i\theta_n x}$$

Or in more compact form:

$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i\theta_n x}$$

Where $C_0 = a_0$, $\quad C_n = \dfrac{a_n + ib_n}{2}$, $n \neq 0$.
$n = \cdots -2, -1, 1, 2, \cdots$

Thus,

$$n \neq 0, \quad C_n = \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx + \frac{i}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx \right]$$

or $$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\theta_n x + i\sin\theta_n x\right)dx$$

So $$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\theta_n x}\,dx$$

Also $$C_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i0x}\,dx$$

Therefore, Fourier Series in its more compact form is given by

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i\theta_n x}, \qquad C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\theta_n x}\,dx \qquad n = \cdots -2, -1, 0, 1, 2, \cdots}$$