3.4 Term by Term Differentiation of Fourier Series.

What is the Fourier Cosine Series of $f(x) \equiv 1$?
over $[0, L]$.

$$1 \sim = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{L} \int_0^L 1\, dx = 1$$

$$a_n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi}{L} x\right) dx = 0, \qquad n = 1, 2, \ldots$$

Then,

$$1 = 1 \quad \checkmark$$

On the other hand, Fourier Sine Series of $f(x) = x$, on $[0, L]$.

$$x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{L} x\right), \qquad 0 \le x < L$$

Jump at $x = L$.

Differentiating term by term,

$$\frac{2\cancel{L}}{\cancel{\pi}} \frac{\cancel{\pi}}{\cancel{L}} \sum \frac{(-1)^{n+1}}{\cancel{n}} \cancel{n} \cos\left(\frac{n\pi}{L} x\right)$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(\frac{n\pi}{L} x\right) \not\equiv 1 \quad ?$$

<u>Thm.</u>– If 1) Fourier Series of $f(x)$ over $[-L, L]$ is Continuous.
2) $f'(x)$ is p.w.s.
Then, F.S. Can be differentiated term by term.

<u>Alternative</u>

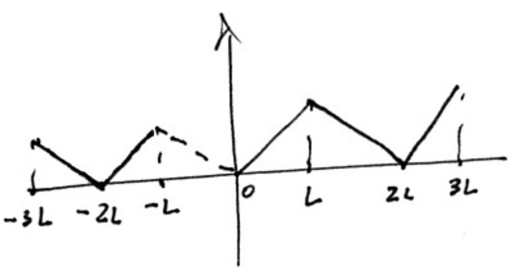
If 1) $f'(x)$ is piecewise smooth on $[-L, L]$
2) $f(x)$ is continuous on $[-L, L]$
3) $f(-L) = f(L)$.
Then, F.S. Can be differentiated term by term.

<u>Apply to Fourier Cosine Series</u>

<u>Prop. 1.</u>– If ① $f'(x)$ is p.w.s. on $[0, L]$
2) $f(x)$ is Cont. on $[0, L]$
Then, F. Can be diff. t. by t.

<u>Example.</u>– $f(x) = x$, on $[0, L]$

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{\substack{n \text{ odd} \\ \text{only}}} \frac{1}{n^2} \cos\left(\frac{n\pi}{L} x\right)$$



Diff. term by term

$$1 \sim \frac{4}{\pi} \sum_{\substack{n \text{ odd} \\ \text{only}}} \frac{1}{n} \sin\left(\frac{n\pi}{L} x\right)$$

Apply to Sine Series.

Prop. 2.– If 1) $f'(x)$ is p.w.s. on $[0,L]$.

2) $f(x)$ is conts. on $[0,L]$.

3) $f(0)=0, \quad f(L)=0$.

Then, Fourier sine series can be differentiated term by term.

Proof.– Since $f(x)$ is conts. on $[0,L]$

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \qquad (3.1)$$

where $B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$.

Also, since $f'(x)$ is p.w.s. on $[0,L]$.

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) \qquad (3.2)$$

where $A_0 = \frac{1}{L}\int_0^L f'(x)dx$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right)dx.$$

If (3.1) were diff. term by term

$$f'(x) \sim \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) B_n \cos\left(\frac{n\pi}{L}x\right)$$

and, comparing with (3.2), it should be verified that

$$A_0 = 0, \qquad A_n = \frac{n\pi}{L} B_n. \qquad (4.1)$$

Equations (4.1) actually hold. In fact,

$$A_0 = \frac{1}{L}\int_0^L f'(x)dx = \frac{1}{L}\left[f(L) - f(0)\right] \overset{\text{Hyp.}}{=} 0 \quad (f(L) = f(0))$$

Also,

$$A_n = \frac{2}{L}\int_0^L f'(x) \cos\left(\frac{n\pi}{L}x\right)dx \overset{\text{IPP}}{=}$$

$$= \frac{2}{L}\left[f(x) \cos\left(\frac{n\pi}{L}x\right)\Big|_0^L + \frac{n\pi}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx \right] =$$

$$= \frac{2}{L}\left[f(L) \cos(n\pi) - f(0)\right] + \frac{n\pi}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left[(-1)^n \underset{}{\overset{\text{Hyp} = 0}{f(L)}} - \overset{\text{Hyp.} = 0}{f(0)}\right] + \frac{n\pi}{L} B_n = \frac{n\pi}{L} B_n. \checkmark$$

We have actually proved that

Prop 3. If 1) $f'(x)$ is p.w.s on $[0, L]$
2) $f(x)$ is conts. on $[0, L]$

Then, if $f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$

$$\Rightarrow f'(x) \sim \frac{1}{L}\left[f(L) - f(0)\right] + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} B_n + \frac{2}{L}\left[(-1)^n f(L) - f(0)\right] \right] \cos\left(\frac{n\pi}{L}x\right).$$