Equilibrium Temperature.

$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}, & 0<x<L,\ t>0. & (1) \\ u(x,0) = f(x), & 0<x<L & (2) \\ u(0,t) = T_1(t), & t>0 & (3) \\ u(L,t) = T_2(t), & t>0 & (4) \end{cases}$$

If BC's were steady $\left(\frac{\partial}{\partial t}=0\right)$ independent of time. We can look for the "equilibrium temperature" or steady-state solution

$\underline{u(x,t) \overset{\text{only}}{=} \hat{u}(x)}$. Thus the above IBVP reduces to a BVP.

$$\begin{cases} \dfrac{d^2\hat{u}}{dx^2} = 0 & (5) \\ \hat{u}(0) = T_1, \quad \hat{u}(L) = T_2 & (6) \end{cases}$$

Gen. soln: $\hat{u}(x) = c_1 x + c_2$

and using B.C's. $\Rightarrow T_1 = u(0) = c_2 \Rightarrow \boxed{c_2 = T_1}$

And $T_2 = \hat{u}(L) = c_1 L + T_1 \Rightarrow \boxed{c_1 = \dfrac{T_2 - T_1}{L}}$

$$\Rightarrow \boxed{\hat{u}(x) = \frac{T_2 - T_1}{L} x + T_1} \qquad (7)$$

The equilibrium temperature distribution does not depend on the I.C.

Physical explanation

Bond. Conds. at the ends will dominate in a long time.

Now, consider IBVP with insulated boundaries:

$$\begin{cases} \dfrac{\partial u}{\partial t} = K \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L. \quad (8) \\ u(x,0) = f(x), & 0 < x < L \quad (9) \\ \dfrac{\partial u}{\partial x}(0,t) = 0, & \dfrac{\partial u}{\partial x}(L,t) = 0 \quad (10) \end{cases}$$

Corresponding equilibrium problem is given by

$$\frac{d^2\hat{u}}{dx^2} = 0, \quad \frac{d\hat{u}}{dx}(0) = 0, \quad \frac{d\hat{u}}{dx}(L) = 0. \qquad (11)$$

So soln. of equil. is

$$\boxed{\hat{u}(x) \equiv C_2}$$

Infinitely many solns.

If we were able to solve the time-dependent problem.

the expectation is that the solution $u(x,t) \xrightarrow[t \to \infty]{} \hat{u}(x) = C_2$

This observation will give us an additional equation from which we will determine c_2.

Consider conservation of energy in the whole rod with c, ρ constants.

$$\frac{d}{dt}\int_0^L e(x,t)\cancel{A}\,dx = \int_0^L c\rho\, u(x,t)\,dx = \phi(0,t)\cancel{A} - \phi(L,t)\cancel{A}$$

$$= -K_0 \overset{=0}{\frac{\partial u}{\partial x}(0,t)} + K_0 \overset{=0}{\frac{\partial u}{\partial x}(L,t)} = 0$$

Then,

$$\text{Total Energy} = E(t) = \int_0^L e(x,t)\,dx = \int_0^L c\rho\, u(x,t)\,dx \equiv \text{constant}$$

$$\Rightarrow \quad E(0) = \lim_{t\to\infty} E(t) \qquad (11')$$

Now, $$E(0) = \int_0^L e(x,0)\,dx = \int_0^L c\rho\, \overbrace{u(x,0)}^{I.C}\,dx = \int_0^L c\rho f(x)\,dx \quad (12)$$

and $$\lim_{t\to\infty} E(t) = \lim_{t\to\infty}\int_0^L c\rho\, u(x,t)\,dx = \int_0^L c\rho \lim_{t\to\infty} u(x,t)\,dx =$$

$$= \int_0^L c\rho\, \overset{\text{Equil. Soln}}{\hat{u}(x)}\,dx = \int_0^L c\rho\, c_2\,dx \qquad (13)$$

Substituting (12) and (13) in (11')

$$\int_0^L \cancel{c\rho}\, f(x)\,dx = \int_0^L \cancel{c\rho}\, C_2\,dx \Rightarrow \int_0^L f(x)\,dx = C_2 L$$

$$\Rightarrow \quad \boxed{C_2 = \frac{1}{L}\int_0^L \underbrace{f(x)\,dx}_{\text{initial condition.}}}$$

Average of initial temperature distribution.