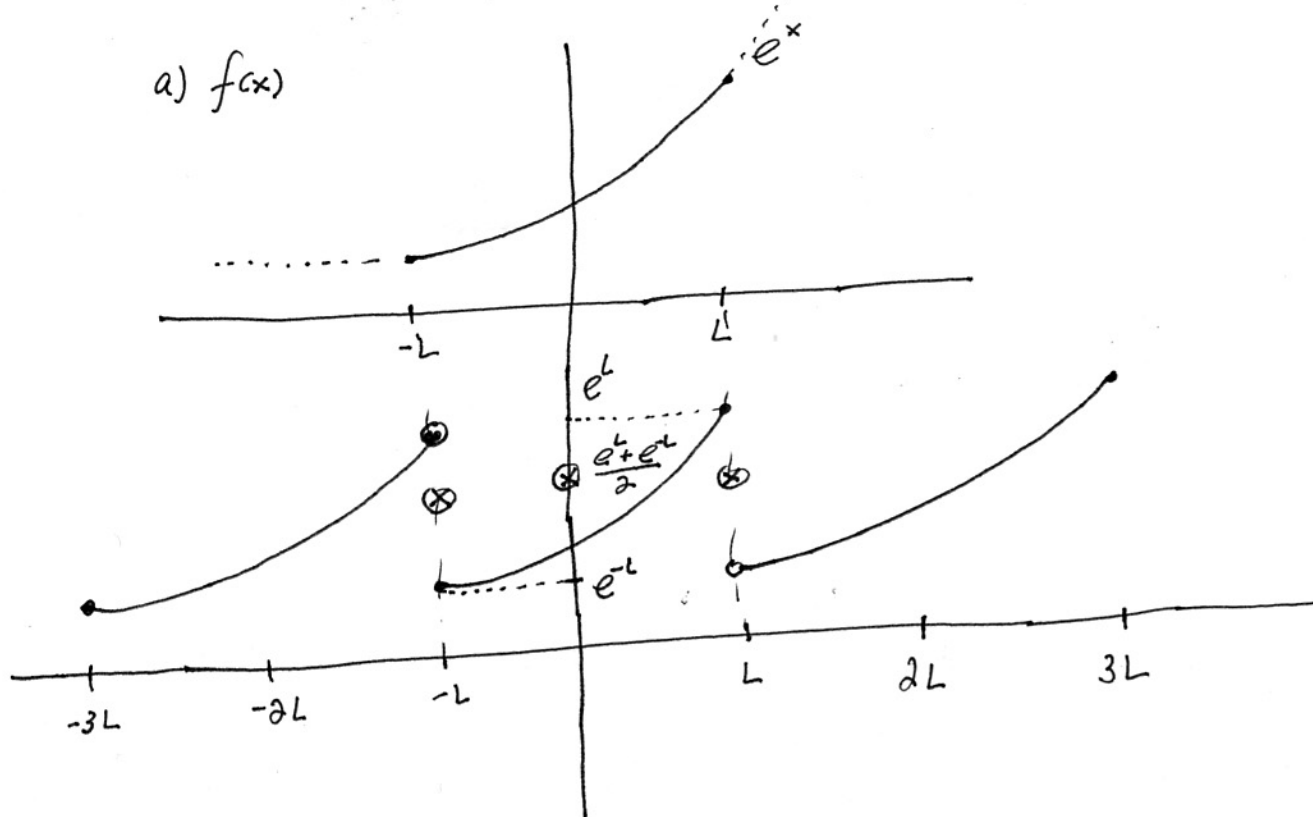


Review of Fourier Series using $f(x) = e^x$.

I) Fourier Series of $f(x) = e^x$, $-L \leq x \leq L$.



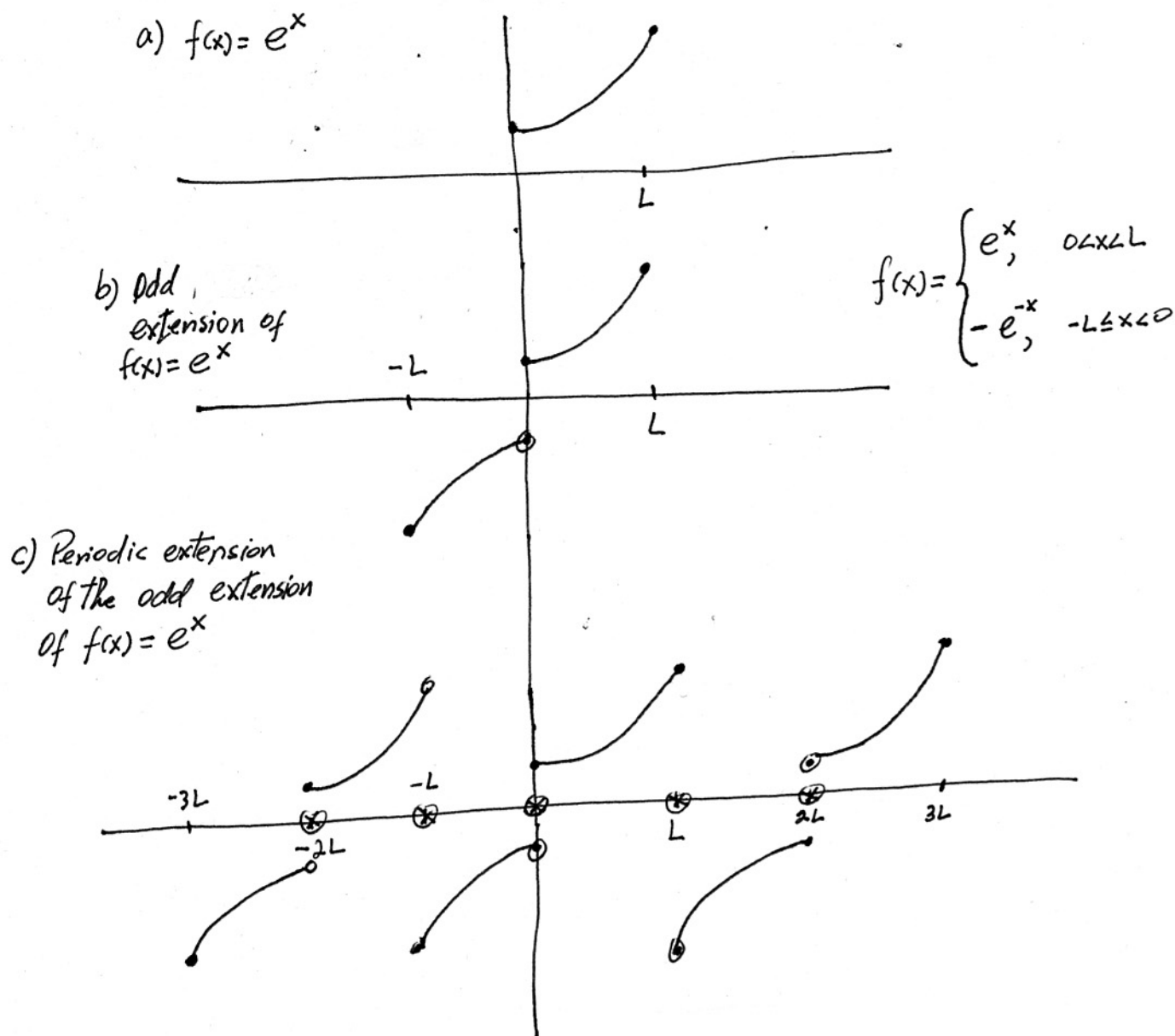
$$e^x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \quad x \in (-\infty, \infty)$$

$$a_0 = \frac{1}{2L} \int_{-L}^L e^x dx = \frac{1}{2L} (e^L + e^{-L})$$

$$a_n = \frac{1}{L} \int_{-L}^L e^x \cos\left(\frac{n\pi}{L}x\right) dx = \frac{(-1)^n L [-1 + e^{2L}]}{L^2 + (n\pi)^2}$$

$$b_n = \frac{1}{L} \int_{-L}^L e^x \sin\left(\frac{n\pi}{L}x\right) dx = \frac{n\pi (-1)^n [1 - e^{2L}]}{L^2 + (n\pi)^2}$$

II) Fourier Series of Sin's of $f(x) = e^x$, $0 \leq x \leq L$.

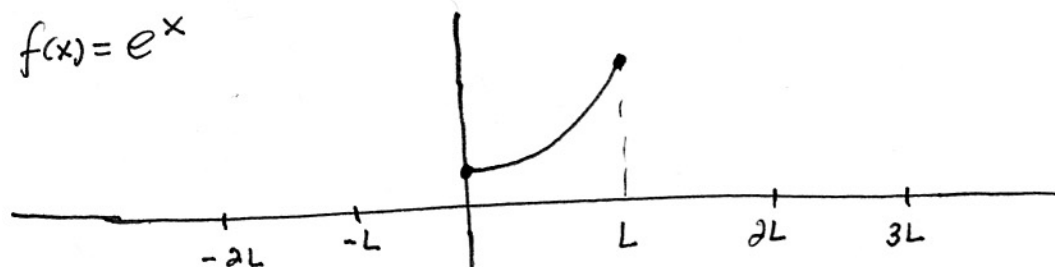


$$e^x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \quad x \in (-\infty, \infty)$$

$$b_n = \frac{2}{L} \int_0^L e^x \sin\left(\frac{n\pi}{L}x\right) dx = \frac{n\pi [1 + (-1)^{n+1} e^L]}{L^2 + (n\pi)^2}$$

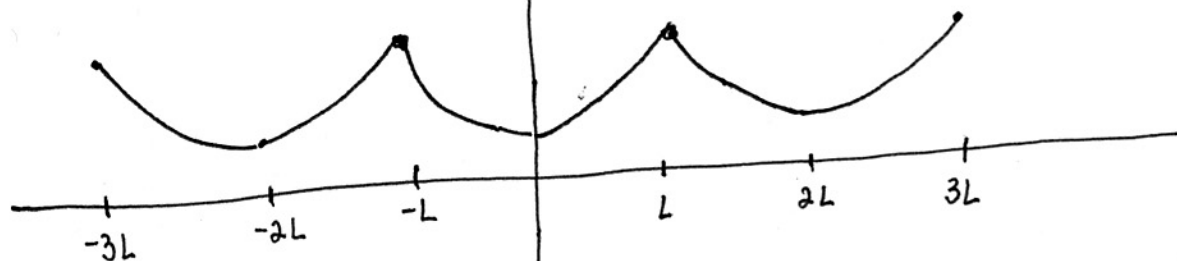
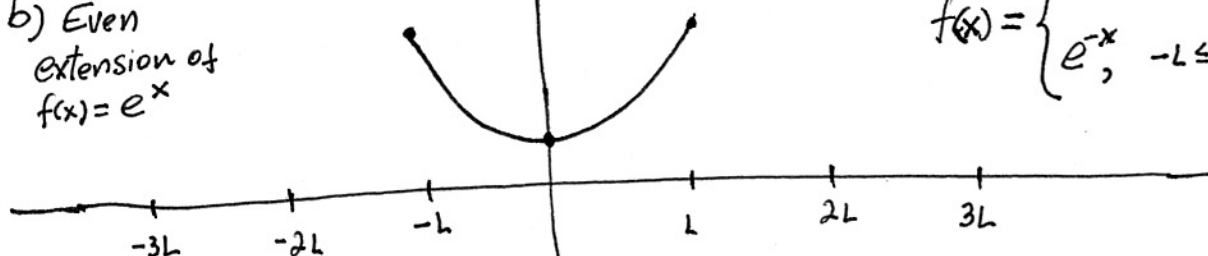
III) Fourier Series of Cosines of $f(x) = e^x$; $0 \leq x \leq L$.

a) $f(x) = e^x$



b) Even extension of $f(x) = e^x$

$$f(x) = \begin{cases} e^x, & 0 \leq x \leq L \\ e^{-x}, & -L \leq x < 0 \end{cases}$$



c)

Periodic extension of the even extension of $f(x) = e^x$.

$$e^x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L} \int_0^L e^x dx = \frac{e^L - 1}{L}$$

$$a_n = \frac{2}{L} \int_0^L e^x \cos\left(\frac{n\pi x}{L}\right) dx = \frac{L [(-1)^n e^L - 1]}{L^2 + (n\pi)^2}$$

3.3.4 Even and odd parts of $f(x)$

Given any function $f(x)$, it's always possible to express $f(x)$ as

$$f(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)] \quad (5.1)$$

Def. - The functions

$$f_o(x) \stackrel{\text{def}}{=} \frac{1}{2} [f(x) + f(-x)] \quad (5.2)$$

$$\text{and } f_e(x) \stackrel{\text{def}}{=} \frac{1}{2} [f(x) - f(-x)] \quad (5.3)$$

are called the even part of $f(x)$ and
the odd part of $f(x)$, respectively.

Thm. - If $f(x)$, $-L \leq x \leq L$ is piecewise smooth,
then

$$f_e(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\text{and } f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where a_n, b_n are the coefficient of the Fourier Series of $f(x)$ over the interval $[-L, L]$.

Proof - Fourier series of $f(x)$ is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow f(-x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) - \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow f_e(x) = \frac{1}{2} [f(x) + f(-x)] \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) \quad (6.1)$$

Even part

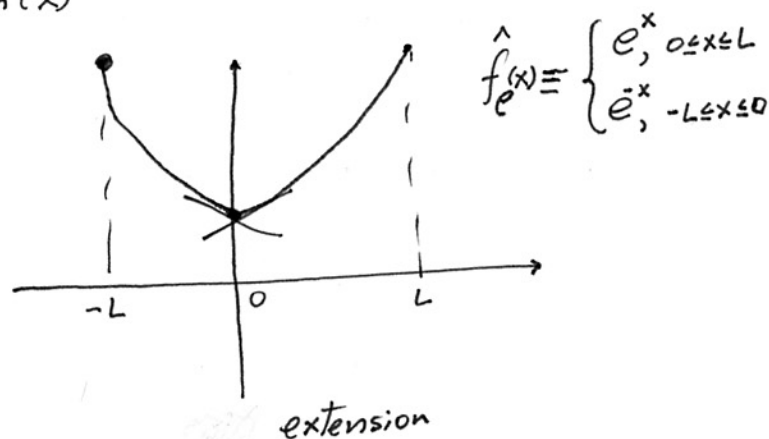
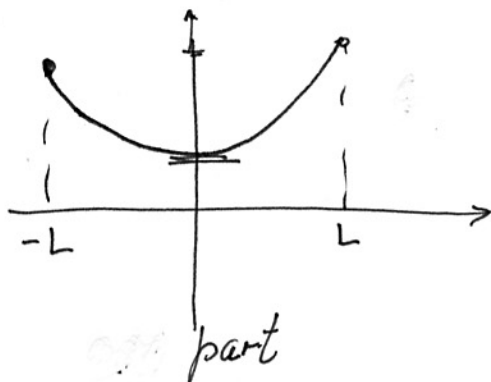
and

$$\Rightarrow f_o(x) = \frac{1}{2} [f(x) - f(-x)] \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \quad (6.2)$$

Odd part

Example: For $f(x) = e^x$, $-L \leq x \leq L$

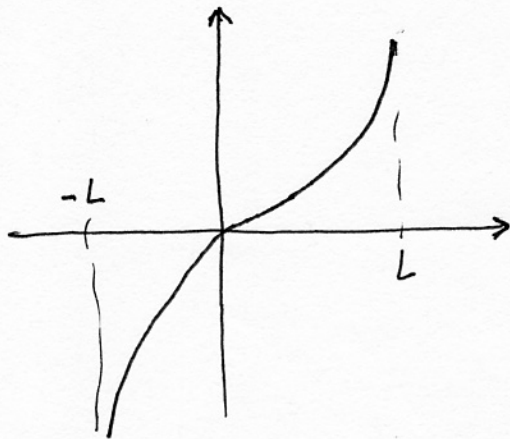
$$f_e(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$



$$\hat{f}_e(x) = \begin{cases} e^x, & 0 \leq x \leq L \\ e^{-x}, & -L \leq x \leq 0 \end{cases}$$

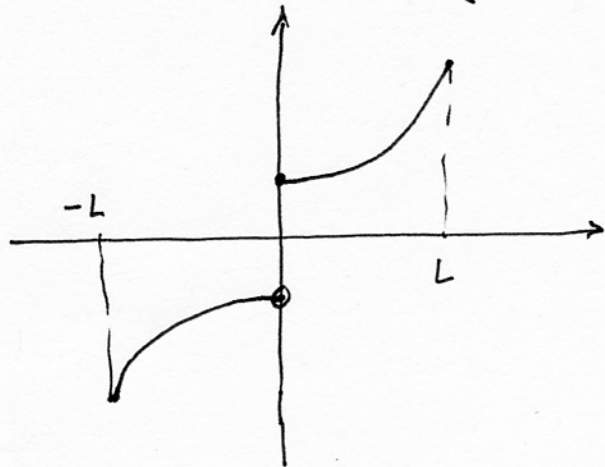
odd part

$$f_o(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$



odd extension

$$\hat{f}_o(x) \equiv \begin{cases} e^x, & 0 \leq x \leq L \\ -e^{-x}, & -L < x < 0 \end{cases}$$



Proposition 1.-

$f(x)$ p.w.s (piecewise smooth) on $[-L, L]$

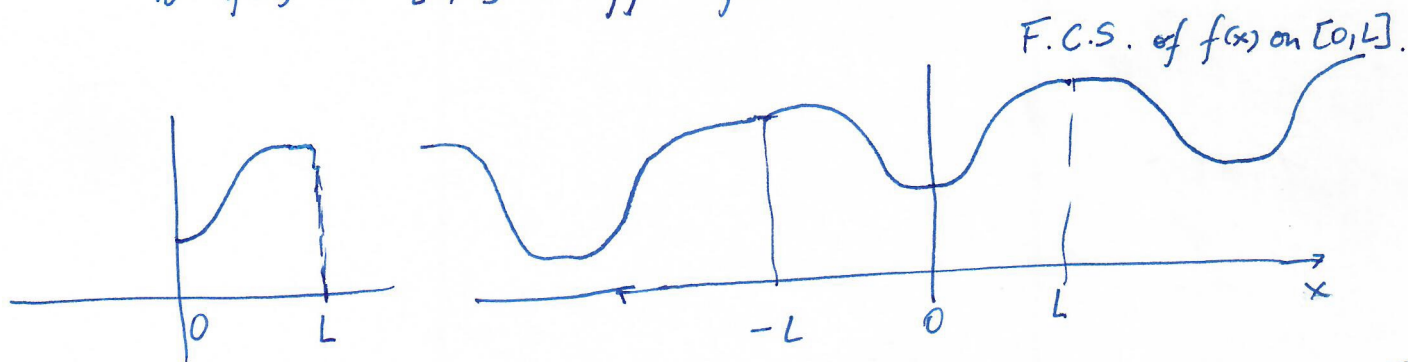
Fourier Series of $f(x)$ is continuous and converges to $f(x)$

on $[-L, L]$ if and only if a) $f(x)$ is cont. on $[-L, L]$.

b) $f(-L) = f(L)$.

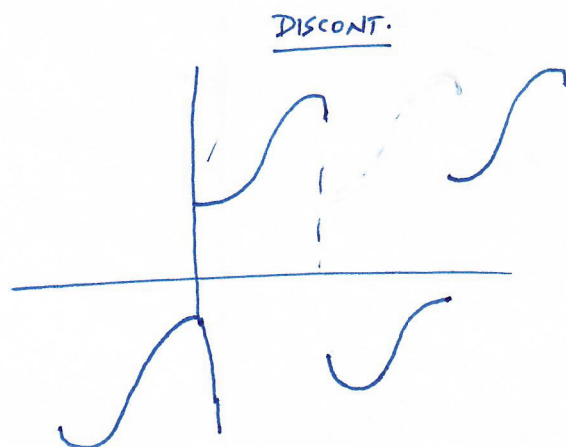
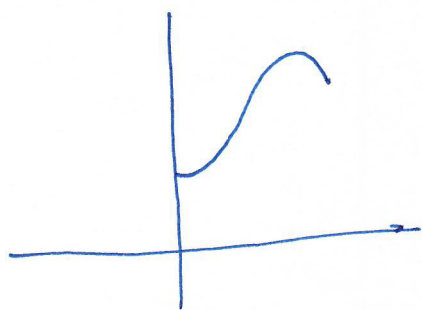
Prop. 2.- $f(x)$ p.w.s. on $[0, L]$.

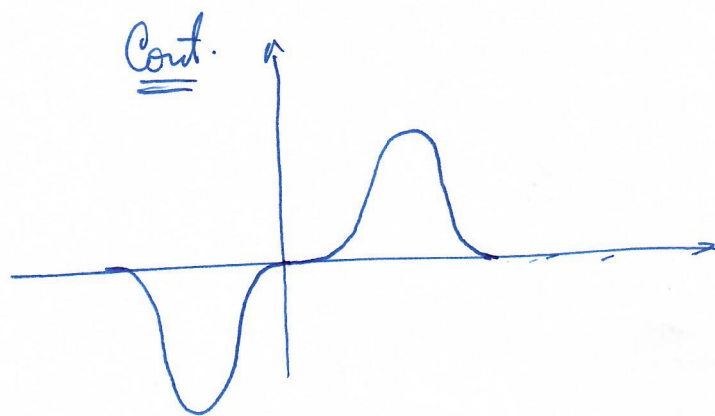
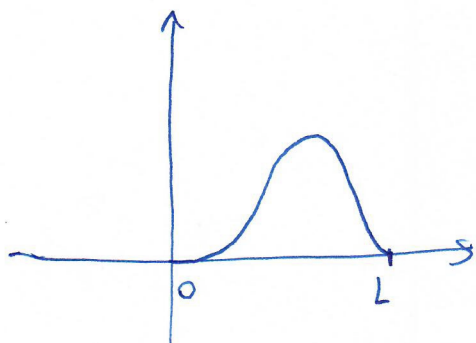
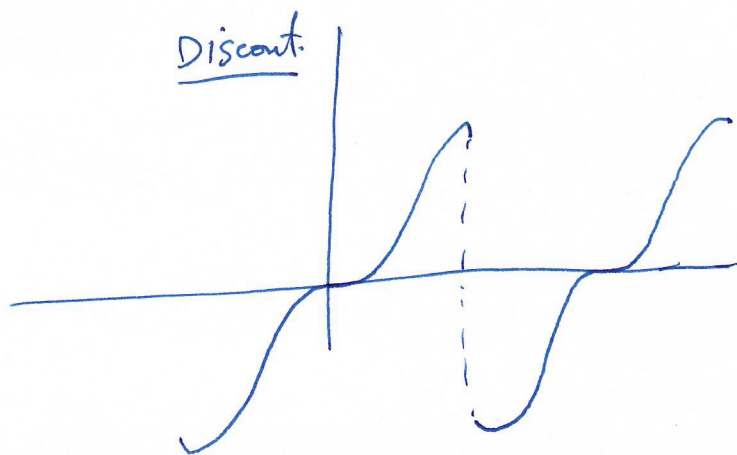
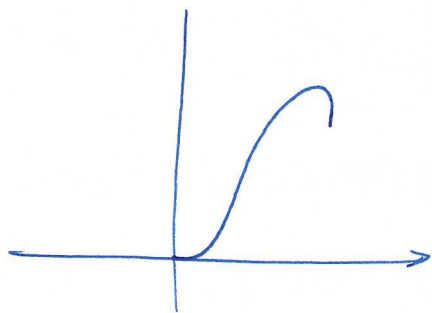
Fourier Cosine Series of $f(x)$ on $[0, L]$ is continuous and converges to $f(x)$ on $[0, L]$ iff $f(x)$ is continuous.



How about

F.S.S. of $f(x)$ on $[0, L]$?





Prop. 3. - $f(x)$ p.w.s. on $[0, L]$.

Fourier Sine Series of $f(x)$ on $[0, L]$ is continuous and converges to $f(x)$ on $[0, L]$ iff $f(x)$ is Conts and both $f(0) = 0$, $f(L) = 0$.