Fourier Series

Given a function $f(x)$ defined on $[-L, L]$

Consider the trigonometric series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \qquad (1)$$

where the coefficients are defined as

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx. \qquad (2)$$

Def.- The Fourier Series of $f(x)$ over the interval $[-L, L]$ is defined to be the infinite series (1), with Fourier coefficients given by (2).

Questions:

1) Does infinite series (1) with coefficients (2) converge?
2) Is F.S. (1) converging to $f(x)$ for every x?
3) what conditions should be imposed on $f(x)$?

- Riemann's Comment on Fourier's first paper.

[1]Fourier series are named for Joseph Fourier, who made the first systematic use, although not a completely rigorous investigation, of them in 1807 and 1811 in his papers on heat conduction. According to Riemann, when Fourier presented his first paper to the Paris Academy in 1807, stating that an arbitrary function could be expressed as a series of the form (1), the mathematician Lagrange was so surprised that he denied the possibility in the most definite terms. Although it turned out that Fourier's claim of generality was somewhat too strong, his results inspired a flood of important research that has continued to the present day. See Grattan-Guinness or Carslaw [Historical Introduction] for a detailed history of Fourier series.

Thm.- (Convergence thm for Fourier Series)

If $f(x)$ is piecewise smooth on the interval $[-L, L]$ then the F.S. of $f(x)$ converges

1.- to the periodic extension of $f(x)$, where the periodic extension is continuous.

2.. to the average of the one side limits

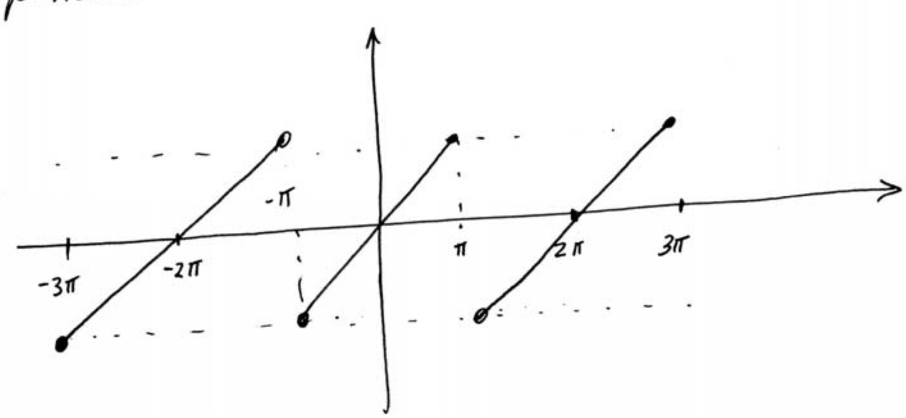
$$\frac{1}{2}\left[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)\right]$$

when the periodic extension has a jump discontinuity at x_0.

Two examples of F.S.

1) $f(x) = x, \qquad -\pi \leq x \leq \pi$

periodic extension



<u>Coefficients:</u>

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{\nearrow \text{odd}}{x} \overset{\nearrow \text{even}}{\cos(nx)}\,dx = 0.$$

$$b_n = \frac{1}{L}\int_{-\pi}^{\pi} x \sin(nx)\,dx = (-1)^{n+1}\frac{2}{n}.$$

Therefore, F.S for $f(x) = x$ on $[-\pi, \pi]$.

$$x \sim \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx) =$$

$$2\left[\sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \cdots\right]$$

2) $$f(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 < x \le 1 \end{cases}$$

F.S. of Sine's only

$$b_n = 2\left[\int_0^{1/2} 2x\sin(n\pi x)\,dx + \int_{1/2}^{1} 2(1-x)\sin(n\pi x)\,dx\right]$$

$$= \frac{8\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}.$$

Therefore, F.S. (Sine's only) of $f(x)$ is given by

$$f(x) \sim \sum_{n=1}^{\infty} \frac{8 \sin(n\pi/2)}{(n\pi)^2} \sin(n\pi x).$$

Fourier Series (Second part).

Nov 9, 2006

Review:

We found the F.S. of

$$\boxed{f(x) = x, \qquad -\pi \le x \le \pi.}$$

The coefficients: a_0, a_n, b_n $\quad n = 1, 2, \ldots$

were determined using the definition of F.S.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x}_{\text{odd}} \underbrace{\cos x}_{\text{even}} \, dx = 0, \qquad n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \, dx = (-1)^{n+1}\frac{2}{n}.$$

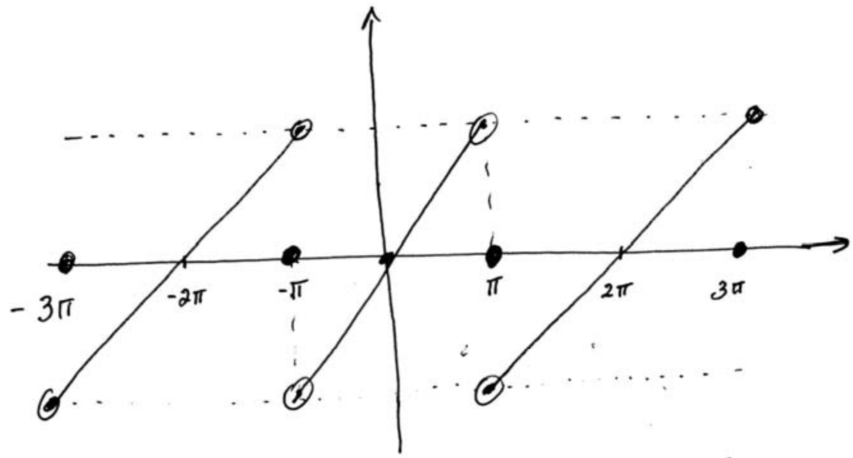
Therefore, F.S. is given by

$$\boxed{\begin{aligned} x &\sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx). = \\ &= 2\left[\sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \cdots\right] \end{aligned}} \qquad (1)$$

F.S. (1) Converges to $f(x) = x$ pointwise on $(-\pi, \pi)$

However, at $x = \pm\pi$ F.S $\rightarrow 0$

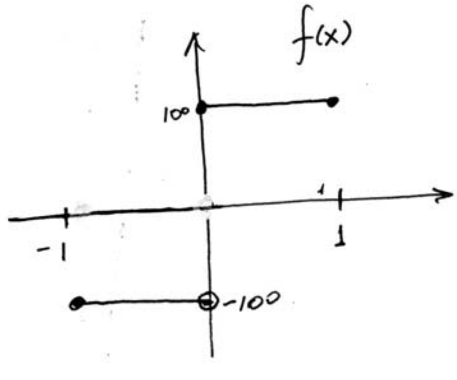
The graph of (1) on $[-3\pi, 3\pi]$ is

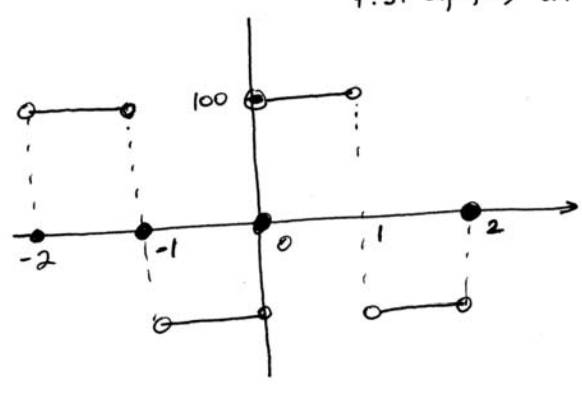


periodic extension of $f(x) = x$, $-\pi \le x \le \pi$.

Example 3.- $f(x) = \begin{cases} 100, & 0 \le x \le 1 \\ -100, & -1 \le x < 0 \end{cases}$

F.S. of $f(x)$ on $[-2,2]$





$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} \overset{\nearrow\text{odd}}{f(x)} \overset{\text{even}}{\cos(n\pi x)}\,dx = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} \overset{\nearrow\text{odd}}{f(x)} \overset{\nearrow\text{odd} \Rightarrow \text{even}}{\sin(n\pi x)}\,dx = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 100\sin(n\pi x)\,dx = \begin{cases} \frac{400}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Therefore,

$$\boxed{f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \sum_{K=1}^{\infty} \frac{400}{\pi}\,\frac{\sin[(2K-1)\pi x]}{2K-1}}$$

<u>Gibbs Phenomenon.</u> overshoot and corresponding undershoot at discontinuities $\simeq 9\%$ of Jump discont.
In our case $\simeq 18$. (9% of 200).