COMPARISON HEAT CONDUCTION IBVP FOR FINITE AND INFINITE DOMAIN. Convolution theorem.

Finite Rod

$$\begin{cases} u_t = k\, u_{xx}, & -L<x<L \\ u(-L,t) = u(L,t) \\ u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$$

Soln:

$$u(x,t) = \sum_{n=0}^{\infty}\left[a_n \cos\tfrac{n\pi}{L}x + b_n \sin\tfrac{n\pi}{L}x\right]e^{-k\left(\frac{n\pi}{L}\right)^2 t} =$$

$$= \sum_{n=-\infty}^{\infty} c_n\, e^{-i\frac{n\pi}{L}x}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} = \boxed{\sum_{n=-\infty}^{\infty} c_n\, e^{-i\omega_n x}\, e^{-k\omega_n^2 t}}$$

$$\omega_n = \frac{n\pi}{L}$$

where $$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi}{L}x}\,dx = \boxed{\frac{1}{2L}\int_{-L}^{L} e^{i\omega_n x} f(x)\,dx}$$

Infinite rod:

$$\begin{cases} u_t = k\, u_{xx}, & -\infty<x<\infty \\ u(-\infty,t) = 0, \quad u(+\infty,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Soln:

$$u(x,t) = \int_{-\infty}^{\infty} c(\omega)\, e^{-i\omega x}\, e^{-k\omega^2 t}\,d\omega \qquad (7.1)$$

Where $$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\,d\bar{x} \qquad (7.2)$$

Solution of Heat Equation on an infinite domain.

$$\begin{cases} u_t = K u_{xx}, & -\infty < x < \infty, \quad t > 0 \quad (5.0) \\ u(x,0) = f(x) & (5.0.1) \\ + \text{B.C's at } \infty : & u(-\infty,t) = 0, \quad u(+\infty,t) = 0 \quad (5.0.2) \end{cases}$$

Separation of variables leads to

$$u(x,t) = \phi(x)\, h(t)$$

(I) $\frac{dh}{dt} + \lambda K h = 0. \Rightarrow h(t) = A e^{-\lambda K t}.$

(II) EVP:
$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda \phi = 0 & (5.1) \\ |\phi(-\infty)| < \infty, \quad |\phi(+\infty)| < \infty & (5.2) \end{cases}$$

Why not $\phi(\infty) = 0, \quad \phi(-\infty) = 0.$?

In order to satisfy (5.1) and (5.2) $\lambda \geq 0$

(If $\lambda < 0$, $\phi(x)$ would be a combination of exponentials and won't be bounded as $x \to \infty$ and $x \to -\infty$ at the same time.)

Thus, $\phi(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$

Satisfies both (5.1) and (5.2) for all $\lambda \geq 0$.

All $\lambda > 0$ are eigenvalues of (5.1) and (5.2)

This is called "continuous spectrum."

Extending the superposition principle from the discrete to the continuous case; we obtain

$$u(x,t) = \int_0^\infty \left[C_1(\lambda)\cos(\sqrt{\lambda}\,x) + C_2(\lambda)\sin(\sqrt{\lambda}\,x)\right] e^{-\lambda k t}\, d\lambda$$

This can be written in terms of ω, if $\lambda = \omega^2$ as

$$\boxed{u(x,t) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] e^{-k\omega^2 t}\, d\omega} \quad (8.1)$$

where

$$A(\omega)\,d\omega = C_1(\omega^2)\,2\omega\,d\omega = C_1(\lambda)\,d\lambda$$

$$B(\omega)\,d\omega = C_2(\omega^2)\,2\omega\,d\omega = C_2(\lambda)\,d\lambda.$$

Using the I.C., we arrive to

$$\boxed{f(x) = u(x,0) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega} \quad (8.2)$$

(8.1) and (8.2) Can be written as (Exercise 10.2.1)

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, e^{-k\omega^2 t}\, d\omega} \quad (9.1)$$

(Inverse Fourier transform of $f(x)$)

and

$$f(x) = u(x,0) = \int_{-\infty}^{\infty} C(\omega)\, e^{-i\omega x}\, d\omega$$

If $f(x)$ is absolutely integrable (plays role of orthogonality),

then

$$\boxed{C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega \bar{x}}\, d\bar{x}} \quad (9.2)$$

(Fourier transform of $f(x)$)

<u>Conclusion:</u>

The Solution of Heat equation in the infinite line with I.C. $u(x,0) = f(x)$ and BCs: $u(\pm\infty, t) = 0$ is given by (9.1)-(9.2)

10.4 <u>Fourier Transform and Heat Equation.</u>

We will Show now that (9.1) and (9.2) Can be combined into a Single integral representation given by

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(\bar{x})\, \frac{1}{\sqrt{4\pi k t}}\, e^{-(x-\bar{x})^2/4kt}\, d\bar{x}} \quad (9.3)$$

(9.1) Can be expressed as

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} \left(C(\omega)\, e^{-\kappa\omega^2 t} \right) e^{-i\omega x}\, dx} \quad (10.1)$$

So $u(x,t)$ can be considered the inverse Fourier transform of the product

$$C(\omega)\, e^{-\kappa\omega^2 t}$$

where (from (9.2)) $C(\omega)$ is the F.T of $f(x)$ (initial condition)

and $e^{-\kappa t\omega^2}$ is the F.T of $g(x) = \sqrt{\frac{\pi}{\kappa t}}\, e^{-x^2/4\kappa t}$.

(from Tables or derivation in Haberman Appendix 10.3)

For the purpose of reducing (10.1) to a single integral, we need the following theorem called "Convolution Theorem".

Theorem (Convolution Theorem)

If

Fourier transform of f(x) — Inverse F.T of F(ω).

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx, \quad f(x) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x} d\omega$$

and
$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\, e^{i\omega x} dx, \quad g(x) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega x} d\omega$$

then, the inverse F.T of the product

$$H(\omega) = F(\omega)\, G(\omega)$$

is
$$\boxed{h(x) = \int_{-\infty}^{\infty} F(\omega)\, G(\omega)\, e^{-i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}}$$

Proof.
$$h(x) = \int_{-\infty}^{\infty} F(\omega)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})\, e^{i\omega\bar{x}} d\bar{x}\right] e^{-i\omega x} d\omega$$

interchanging order of integration
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})\left[\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega(x-\bar{x})} d\omega\right] d\bar{x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})\, f(x-\bar{x})\, d\bar{x}$$

If we substitute F(ω) instead of G(ω), then

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}.$$

Application of convolution theorem.

In (10.1)

$C(\omega)$ is F.T of $f(x)$ (initial cond.)

$e^{-\kappa\omega^2 t}$ is F.T of $g(x) = \sqrt{\frac{\pi}{\kappa t}}\, e^{-\frac{x^2}{4\kappa t}}$ (From Table)

Then using convolution theorem

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) \sqrt{\frac{\pi}{\kappa t}}\, e^{-\frac{(x-\bar{x})^2}{4\kappa t}}\, d\bar{x}$$

or

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi\kappa t}}\, e^{-\frac{(x-\bar{x})^2}{4\kappa t}}\, d\bar{x}}$$