10.4.2 Transforms of Derivatives. Application to Heat Conduction

The Fourier transforms provide a valuable tool to solve PDE. For this, we need to know how to compute the F.T. of

$$\frac{\partial u}{\partial t}, \quad \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^4 u}{\partial x^4}, \text{ etc..}$$

Let's begin with the F.T. of $\frac{\partial u}{\partial x}(x,t)$.

Theorem The F.T. of the first derivative of $u(x,t)$ with respect to x equals $-i\omega \bar{U}(\omega,t)$, assuming that $u(x,t) \xrightarrow[x\to\pm\infty]{} 0$ sufficiently fast. and $u_x(x,t)$ is continuous.

Proof: Let $\bar{U}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\, dx$, t fixed.

then,

$$\mathcal{F}\left[\frac{\partial u}{\partial x}(x,t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,t)\, e^{i\omega x}\, dx$$

$$\overset{\text{I.B.P}}{=} \frac{1}{2\pi}\left[u(x,t)\, e^{i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\omega\, u(x,t)\, e^{i\omega x}\, dx\right]$$

Using that $u(x,t) \xrightarrow[x \to \pm\infty]{} 0$ then $u(x,t)\, e^{i\omega x} \xrightarrow[x \to \pm\infty]{} 0$

$$\mathcal{F}\left[\frac{\partial u}{\partial x}(x,t)\right] = -i\omega\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\, dx\right]$$

or

$$\boxed{\mathcal{F}\left[\frac{\partial u}{\partial x}(x,t)\right] = -i\omega\, \bar{U}(\omega,t)} \qquad (2.1)$$

<u>Theorem.</u> If $u(x,t)$ is C^2 in x and $u(x,t) \xrightarrow[x \to \pm\infty]{} 0$ sufficiently fast and $u_x(x,t) \xrightarrow[x \to \pm\infty]{} 0$ then the F.T of the second derivative w.r.t. x is

$$\boxed{\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}(x,t)\right] = -\omega^2\, \bar{U}(\omega,t)} \qquad (2.2)$$

<u>Proof.</u>

$$\mathcal{F}[u_{xx}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} (u_x)_x\, e^{i\omega x}\, dx =$$

$$= \frac{1}{2\pi}\left[\cancelto{0}{u_x e^{i\omega x}\Big|_{-\infty}^{\infty}} - i\omega\int_{-\infty}^{\infty} u_x\, e^{i\omega x}\, dx\right]$$

$$= -i\omega\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u_x\, e^{i\omega x}\, dx\right) = -i\omega\, \mathcal{F}[u_x] =$$

$$= (-i\omega)(-i\omega)\, \bar{U}(\omega,t) = -\omega^2\, \bar{U}(\omega,t) \checkmark$$

In general,

<u>Theorem.</u> If $u(x,t)$ is C^n in x and

$\frac{\partial^n u}{\partial x^n} \underset{x\to\pm\infty}{\longrightarrow} 0$, $n=0,1,\dots$, then the F.T of the $\underset{\text{w.r.t. } x}{\frac{\partial^n u}{\partial x^n}}$

is
$$\boxed{\mathcal{F}\left[\frac{\partial^n u}{\partial x^n}\right] = (-i\omega)^n \bar{U}(\omega,t).} \quad (3.1)$$

We also need to find the F.T of $\frac{\partial u}{\partial t}(x,t)$

With respect to x

<u>Theorem.</u> If $\frac{\partial u}{\partial t}(x,t)$ is C' in x, then F.T w.r.t. x

$$\boxed{\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t}\bar{U}(\omega,t).} \quad (3.2)$$

<u>Proof.</u>
$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\, e^{i\omega x}\, dx$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\, dx\right] = \frac{\partial \bar{U}}{\partial t}(\omega,t).$$

Similarly,
$$\underset{\text{F.T w.r.t. } x}{\boxed{\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2 \bar{U}}{\partial t^2}(\omega,t).}} \quad (3.3)$$

Alternative derivation of Solution for infinite BVP for Heat Equation, using transforms of derivatives.

Infinite Rod:

$$\begin{cases} u_t = k\, u_{xx}, \quad -\infty < x < \infty & (4.1) \\ u(-\infty, t) = 0, \quad u(+\infty, t) = 0 & (4.2) \\ u(x, 0) = f(x) & (4.3) \end{cases}$$

Applying F.T. w.r.t. x to the two sides of (4.1)

$$\boxed{\frac{\partial \bar{U}}{\partial t}(\omega, t) = -\omega^2 k\, \bar{U}(\omega, t)} \qquad (4.4)$$

where $\bar{U}(\omega, t) = \mathcal{F}[u(x,t)]$ with respect to x.

Equation (4.4) is a first order ODE for the time variable

then $$\boxed{U(\omega, t) = C(\omega)\, e^{-k\omega^2 t}} \qquad (4.5)$$

Applying F.T. to (4.3)

$$\bar{U}(\omega, 0) = F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \quad (4.6)$$

From (4.5) $U(\omega, 0) = C(\omega) \Rightarrow C(\omega) = F(\omega)$

Therefore, $$\boxed{U(\omega, t) = F(\omega)\, e^{-k\omega^2 t}} \qquad (4.7)$$

We can now apply the convolution theorem to (4.7)

Knowing that

a) $f(x)$ is inverse F.T. de $F(\omega)$

and b) $\sqrt{\frac{\pi}{\kappa t}}\, e^{-x^2/4\kappa t}$ is the inverse F.T. of $e^{-\kappa\omega^2 t}$. (From Tables)

Therefore,

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\sqrt{\frac{\pi}{\kappa t}}\, e^{-(x-\bar{x})^2/4\kappa t}\, d\bar{x}$$

or

$$\boxed{U(x,t) = \int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi\kappa t}}\, e^{-(x-\bar{x})^2/4\kappa t}\, d\bar{x}} \quad (5.1)$$