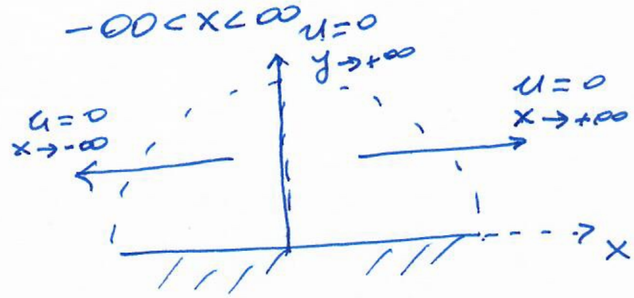
10.6.3 Laplace's Equation in a Half-Plane.

Consider

$$\begin{cases} u_{xx} + u_{yy} = 0, & (1.1) \\ \text{B.C.: } u(x,0) = f(x) & (1.2) \end{cases}$$



So other BC's are

$$u(+\infty, y) = 0 \qquad (1.3)$$

$$u(-\infty, y) = 0 \qquad (1.4)$$

$$u(x, +\infty) = 0 \qquad (1.5)$$

The two homogeneous B.C's in "x" (1.3) and (1.4) suggests to take F.T. in "x" with "y" fixed.

F.T of $u(x,y)$ with respect to "x"

$$\bar{U}(\omega, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,y)\, e^{i\omega x}\, dx, \quad y \text{ fixed.}$$

and

$$u(x,y) = \int_{-\infty}^{\infty} \bar{U}(\omega, y)\, e^{-i\omega x}\, d\omega$$

Applying F.T. to the individual terms of (1.1)

$$\mathcal{F}[u_{xx}(x,y)] \overset{\text{Previous Thm}}{=} -\omega^2 \bar{U}(\omega,y)$$

$$\mathcal{F}[u_{yy}(x,y)] = \frac{\partial^2}{\partial y^2}\bar{U}(\omega,y) = \bar{U}_{yy}(\omega,y)$$

Since, $\mathcal{F}[u_{yy}(x,y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_{yy}(x,y)\, e^{i\omega x}\,dx$

interchanging derivatives and integral order $= \frac{\partial^2}{\partial y^2}\left[\int_{-\infty}^{\infty} u(x,y)\, e^{i\omega x}\,dx\right] = \bar{U}_{yy}(\omega,y)$

Therefore, (1.1) is reduced to

$$\boxed{\bar{U}_{yy}(\omega,y) - \omega^2 \bar{U}(\omega,y) = 0} \qquad (2.1)$$

Whose general solution is given by

$$\boxed{\bar{U}(\omega,y) = a(\omega)\, e^{\omega y} + b(\omega)\, e^{-\omega y}} \qquad (2.2)$$

with $a(\omega)$ and $b(\omega)$ to be determined from B.Cs.

Applying F.T. to B.C. (1.2).

$$\bar{U}(\omega,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)\, e^{i\omega x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\,dx = F(\omega)$$

or $$\boxed{\bar{U}(\omega,0) = F(\omega).} \qquad (2.3)$$

Also, applying F.T. to B.C. (1.5)

$$u(x,+\infty)=0 \;\Rightarrow\; \bar{U}(\omega,+\infty)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overset{=0}{u(x,+\infty)}\,e^{i\omega x}\,dx = 0.$$

As a consequence, the general solution (2.2)

is reduced to

$$\bar{U}(\omega,y)=\begin{cases} a(\omega)\,e^{\omega y}, & \omega<0 \\ b(\omega)\,e^{-\omega y}, & \omega>0 \end{cases}$$

Or in more compact form:

$$\boxed{\bar{U}(\omega,y)=C(\omega)\,e^{-|\omega| y}} \qquad (3.1)$$

where $$C(\omega)=\begin{cases} a(\omega), & \omega<0 \\ b(\omega), & \omega>0 \end{cases}$$

But, also $$C(\omega)=\bar{U}(\omega,0)\overset{(2.3)}{=}F(\omega)$$

Therefore, $$\boxed{\bar{U}(\omega,y)=F(\omega)\,e^{-|\omega| y}} \qquad (3.2)$$

product of two F.T.

If we know the inverse F.T. of $F(\omega)$ and $e^{-|\omega| y}$ then by application of the convolution theorem we can obtain the solution $u(x,y)$ sought.

The inverse F.T. of $F(\omega)$ is $f(x)$.

but the inverse F.T. of $e^{-|\omega|y}$ is unknown.

The good news is that it can be computed directly. In fact,

$$\mathcal{F}^{-1}[e^{-|\omega|y}] = \int_{-\infty}^{\infty} e^{-|\omega|y} e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{0} e^{\omega y} e^{-i\omega x}\, d\omega + \int_{0}^{\infty} e^{-\omega y} e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{0} e^{\omega(y-ix)}\, d\omega + \int_{0}^{\infty} e^{-\omega(y+ix)}\, d\omega$$

$$= \frac{e^{\omega(y-ix)}}{y-ix}\Bigg|_{\omega=-\infty}^{\omega=0} + \frac{e^{-\omega(y+ix)}}{-(y+ix)}\Bigg|_{\omega=0}^{\omega=+\infty}$$

$$e^{-\omega(y+ix)} = e^{-\omega y} \underbrace{e^{-i\omega x}}_{\text{bounded}} \xrightarrow[\omega\to\infty]{} 0$$

$$= \frac{1}{y-ix} + \frac{1}{y+ix} = \frac{2y}{x^2+y^2}$$

It means

$$\boxed{\mathcal{F}^{-1}[e^{-|\omega|y}] = \frac{2y}{x^2+y^2}.} \qquad (4.1)$$

Application of the convolution theorem:

$$\bar{U}(\omega, y) \overset{(3.2)}{=} F(\omega)\, e^{-|\omega| y}$$

where $f(x)$ is the inverse F.T. of $F(\omega)$

and $\dfrac{2y}{x^2+y^2}$ is the inverse F.T. of $e^{-|\omega| y}$

then,

$$\boxed{u(x,y) \overset{\text{Convolution Theorem}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \, \frac{2y}{(x-\bar{x})^2 + y^2} \, d\bar{x}} \qquad (5.1)$$

inverse F.T of $\bar{U}(\omega, y)$