10.5 Fourier Sine and Cosine Transforms.

If $f(x)$ is odd the Fourier Sine Transform Pair is

Inverse Fourier Sine Transform :
$$f(x) = \int_0^{\infty} F_s(\omega) \sin(\omega x) d\omega$$

Fourier Sine Transform :
$$F_s(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin(\omega x) dx$$

If $f(x)$ is even the Fourier Cosine Transform Pair

I. F. Cosine Transform :
$$f(x) = \int_0^{\infty} F_c(\omega) \cos \omega x \, d\omega$$

F. Cosine Transform :
$$F_c(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos \omega x \, dx$$

Fourier Sine/cosine transforms of derivatives.

We will adopt this notation:

$$\boxed{C[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx = F_c(\omega)} \quad (2.1)$$

and

$$\boxed{S[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx = F_s(\omega)} \quad (2.2)$$

Using integration by parts, we will obtain the Fourier Sine/Cosine transforms of $\frac{df}{dx}(x)$.

Theorem

If $f(x)$ is conts. and $f(x) \underset{x\to\pm\infty}{\longrightarrow} 0$

Then $$\boxed{C\left[\frac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega S[f]} \quad (2.3)$$

Proof.

$$C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^\infty \frac{df(x)}{dx}\cos\omega x\, dx \overset{I.P.P.}{=}$$

$$\frac{2}{\pi}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x\, dx\right] =$$

$$+\frac{2}{\pi}\overset{0}{f(\infty)}\overset{\text{bded}}{\cos\omega(\infty)} - \frac{2}{\pi}f(0) + \omega S[f] =$$

$$-\frac{2}{\pi}f(0) + \omega S[f]$$

Theorem. If $f(x)$ is conts. and $f(x) \xrightarrow[x \to \pm\infty]{} 0$

then,

$$\boxed{S\left[\frac{df}{dx}\right] = -\omega C[f].} \qquad (3.1)$$

Proof.

$$S\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^\infty \frac{df}{dx}(x)\sin\omega x\, dx$$

$$= \frac{2}{\pi}\left[f(x)\sin\omega x\Big|_0^\infty - \omega\int_0^\infty f(x)\cos\omega x\, d\omega\right]$$

$$= -\omega C[f]$$

Theorem If $f(x)$ is conts, $\frac{df}{dx}$ is conts, $f, \frac{df}{dx} \xrightarrow[x \to \pm\infty]{} 0$.

Then,

a) $$\boxed{C\left[\frac{d^2f}{dx^2}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 C[f]} \qquad (3.2)$$

b) $$\boxed{S\left[\frac{d^2f}{dx^2}\right] = \frac{2}{\pi}\omega f(0) - \omega^2 S[f].} \qquad (3.3)$$

Proof.

a) $$C\left[\frac{d^2f}{dx^2}\right] \overset{(2.3)}{=} -\frac{2}{\pi}\frac{df}{dx}(0) + \omega S\left[\frac{df}{dx}\right]$$

$$\overset{(3.1)}{=} -\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 C[f]$$

b) $S\left[\frac{d^2f}{dx^2}\right] \overset{(3.1)}{=} -\omega C\left[\frac{df}{dx}\right]$

$$\overset{(2.3)}{=} \left(-\frac{2}{\pi}\frac{df}{dx}(0) + \omega S[f]\right)(-\omega)$$

$$= \frac{2}{\pi}\omega\frac{df}{dx}(0) - \omega^2 S[f].$$

This formulas can be easily extended to Fourier cosine/sine transforms of $\frac{d^n f}{dx^n}$. $n = 3, 4, \ldots N$

Remark:

a) Application of F. cosine transform requires $\frac{df}{dx}(0)$ to be known (Neumann condition.)

b) " " F. sine transform requires $f(0)$ to be known (Dirichlet B.C.).

Application of Fourier Sine/Cosine transform to Heat Equation in a Semi-infinite interval.

$$\begin{cases} u_t = k u_{xx}, \quad 0 < x < \infty, \quad t > 0. & (5.1) \\ \text{B.C.} \quad u(0,t) = g(t), & (5.2) \\ \text{I.C.} \quad u(x,0) = f(x) & (5.3) \end{cases}$$

If $g(t) \not\equiv 0$, this problem has nonhomogeneous B.C. different than previously analized problems.

Solution:

Applying F. Sine transform $\left(\begin{array}{c}\text{guided by the} \\ \text{Dirichlet B.C.}\end{array}\right)$

$$\bar{U}_t(\omega,t) = k\,S[u_{xx}]$$

$$= \frac{2k}{\pi}\omega\, u(0,t) - k\omega^2 S[u] = \frac{2}{\pi}k\omega\, g(t) - \omega^2 k \bar{U}(\omega,t)$$

where $\bar{U}(\omega,t) = \frac{2}{\pi}\int_0^\infty u(x,t)\sin\omega x\, dx = S[u]$

So we obtain the 1st order linear nonhomogeneous ODE:

$$\boxed{\bar{U}_t(\omega,t) + k\omega^2 \bar{U}(\omega,t) = \frac{2k}{\pi}\omega\, g(t)} \qquad (5.4)$$

Equ. (5.4) Can be easily solve using an integrating factor. for this ODE.

To simplify the computation, let us assume $g(t) \equiv 0$,
then

$$\boxed{\bar{U}(\omega,t) = C(\omega)\, e^{-k\omega^2 t}} \qquad (6.1)$$

Applying F. Sine transform to I.C. (5.3)

$$\bar{U}(\omega,0) = F_s(\omega) = \frac{2}{\pi}\int_0^\infty f(x) \sin \omega x \, dx$$

Also from (6.1)

$$\bar{U}(\omega,0) = C(\omega) \quad \text{then} \quad \boxed{C(\omega) = F_s(\omega)} \quad (6.2)$$

and Substitution into (6.1) leads to

$$\boxed{\bar{U}(\omega,t) = F_s(\omega)\, e^{-k\omega^2 t}} \qquad (6.3)$$

At this point, we will introduce a convolution for Fourier Sine/cosine transforms.

Then, we will apply it to (6.3) to obtain the Solution $u(x,t)$.

Fourier Sine/cosine transform Convolution Theorem.

Theorem Assume

$\bar{S}(\omega)$ is the Sine transform of $S(x)$.

$\bar{C}(\omega)$ is the cosine transform of $C(x)$.

and $S(x)$ is odd and $C(x)$ is even.

Also, assume that

$$H(\omega) = \bar{S}(\omega)\,\bar{C}(\omega).$$

Then,

$$h(x) = \frac{1}{\pi}\int_0^{\infty} S(\bar{x})\left[C(x-\bar{x}) - C(x+\bar{x})\right]d\bar{x}$$

$$= \frac{1}{\pi}\int_0^{\infty} C(\bar{x})\left[S(x+\bar{x}) - S(\bar{x}-x)\right]d\bar{x}$$

Proof.

Back to the solution of (5.1) − (5.3)

We already have obtained

$$\bar{U}(\omega,t) = F_s(\omega)\, e^{-k\omega^2 t}$$

where

$F_s(\omega)$ is the <u>Sine F.T.</u> of $f(x)$

and $e^{-k\omega^2 t}$ is the <u>Cosine F.T.</u> of $\frac{1}{2}\sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}$

(Table 10.5.2 in book)

In fact,

$$\frac{1}{\sqrt{\pi\alpha}}\, e^{-\omega^2/4\alpha} \text{ is F. cosine transform of } e^{-\alpha x^2}.$$

Therefore, if $\frac{1}{4\alpha} = kt \Rightarrow \alpha = \frac{1}{4kt}$

and

$$\frac{1}{\sqrt{\pi\alpha}}\, e^{-\omega^2/4\alpha} = \frac{2\sqrt{kt}}{\sqrt{\pi}}\, e^{-kt\omega^2} \xrightarrow[\text{of}]{\text{F.C.T.}} e^{-\frac{x^2}{4kt}}$$

$$\text{or } e^{-kt\omega^2} \xrightarrow[\text{of}]{\text{F.C.T}} \frac{1}{2}\sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt}$$

Therefore, applying conv. theorem in SC_7

$$u(x,t) = \frac{1}{\pi}\int_0^\infty f(\bar{x})\, \frac{1}{2}\sqrt{\frac{\pi}{kt}}\left[e^{-\frac{(x-\bar{x})^2}{4kt}} - e^{-\frac{(x+\bar{x})^2}{4kt}}\right] d\bar{x}$$

or

$$\boxed{u(x,t) = \int_0^\infty f(\bar{x})\frac{1}{\sqrt{4\pi kt}}\left[e^{-\frac{(x-\bar{x})^2}{4kt}} - e^{-\frac{(x+\bar{x})^2}{4kt}}\right] d\bar{x}}$$