

10.6.5

2-D Fourier Transform. Heat Equation

$$\begin{cases} u_t = \kappa (u_{xx} + u_{yy}) & -\infty < x, y < \infty, t > 0 \\ u(x, y, 0) = f(x, y) \\ u(x, y, t) \rightarrow 0, \text{ when } x, y \rightarrow \pm\infty. \end{cases}$$

Sep of Vars. lead to

$$u(x, y, t) = \phi(x, y) h(t)$$

$$h'(t) = -\lambda \kappa h(t) \Rightarrow \boxed{h(t) = C e^{-\lambda \kappa t}}$$

$$\begin{cases} \nabla^2 \phi + \lambda \phi = 0 \\ |\phi(x, y)| < \infty, \text{ when } x, y \rightarrow \pm\infty \end{cases}$$

$$\text{Further Sep. : } \phi(x, y) = V(x) \omega(y)$$

$$V'' \omega + V \omega'' + \lambda V \omega = 0$$

 $\frac{1}{V\omega}$

$$\frac{V''}{V} + \frac{\omega''}{\omega} + \lambda = 0 \Rightarrow \frac{V''}{V} + \lambda = -\frac{\omega''(y)}{\omega(y)} = +\mu$$

$$\Rightarrow V'' + (\lambda - \mu)V = 0$$

$$\omega'' + \mu\omega = 0$$

$$\Rightarrow \mu > 0 \quad \omega(y) = e^{-i\omega_2 y} \quad \omega_2 = \pm\sqrt{\mu}$$

$$V_{\lambda, \mu}(x) = e^{-i\omega_1 x}$$

$$\omega_1 = \pm\sqrt{\lambda - \mu} > 0$$

$$\Rightarrow \lambda = \omega_1^2 + \omega_2^2$$

Therefore,

$$\boxed{\phi(x, y) = e^{-i\omega_1 x} e^{-i\omega_2 y}}, \quad \omega_1, \omega_2 \in \mathbb{R}$$

Therefore, the product soln:

$$U(x, y, t) = e^{-i\omega_1 x} e^{-i\omega_2 y} e^{-\kappa(\omega_1^2 + \omega_2^2)t}$$

Using Superposition principle with continuous spectrum

$$U(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega_1, \omega_2) e^{-i\omega_1 x} e^{-i\omega_2 y} e^{-\kappa(\omega_1^2 + \omega_2^2)t} d\omega_1 d\omega_2 \quad (2.0)$$

The initial condition implies

$$f(x, y) = U(x, y, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{F}(\omega_1, \omega_2) e^{-i\omega_1 x} e^{-i\omega_2 y} d\omega_1 d\omega_2 \quad (2.0.1)$$

It can be easily shown that

$$\tilde{F}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i\omega_1 x} e^{i\omega_2 y} dx dy \quad (2.0.2)$$

if $f(x, y)$ has Fourier Transf. in each variable x and y .

Proof:- First consider the F.T of f w.r.t. x (y fixed)

$$F(\omega_1, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x, y) e^{i\omega_1 x} dx \quad (2.1)$$

and its inverse

$$f(x, y) = \int_{-\infty}^{\infty} F(\omega_1, y) e^{-i\omega_1 x} d\omega_1 \quad (2.2)$$

Since $F(\omega, y)$ is also a function of "y", it is possible to find its F.T.

$$\tilde{F}(\omega_1, \omega_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega_1, y) e^{i\omega_2 y} dy \quad (3.1)$$

and
$$F(\omega, y) = \int_{-\infty}^{\infty} \tilde{F}(\omega_1, \omega_2) e^{-i\omega_2 y} d\omega_2 \quad (3.2)$$

Combining (2.1) - (2.2) with (3.1) - (3.2).

2-D Fourier Transform Pair

$$\tilde{F}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i\omega_1 x} e^{i\omega_2 y} dx dy \quad (3.3)$$

Same as (2.0.1)

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{F}(\omega_1, \omega_2) e^{-i\omega_2 y} e^{-i\omega_1 x} d\omega_2 d\omega_1 \quad (3.4)$$

Same as (2.0).

Alternative notation introducing the "wave number vector."

$$e^{i\omega_1 x} e^{i\omega_2 y} = e^{i(\omega_1 x + \omega_2 y)} = e^{i\vec{\omega} \cdot \vec{r}}$$

where $\vec{r} = (x, y), \quad \vec{\omega} = (\omega_1, \omega_2)$

The lines $\omega_1 x + \omega_2 y = \text{const}$ are called wave front
the normal direction to this lines is in the direction of the gradient

$$\nabla(\omega_1 x + \omega_2 y) = (\omega_1, \omega_2)$$

Thus, this vector (w_1, w_2) is the direction of propagation of the wave.

The magnitude $\omega = |\vec{\omega}|$

is the number of waves (full period) in 2π distance along the direction of propagation.
 This is why $\vec{\omega}$ is called "wave number vector"
 And $\omega = |\vec{\omega}|$ the "wave number"

Using this notation the pair (3.3)-(3.4) can be written as

$$F(\vec{\omega}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{r}) e^{i\vec{\omega} \cdot \vec{r}} d\vec{r} \quad \underbrace{d\vec{r} = dx dy}$$

$$f(\vec{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\vec{\omega}) e^{-i\vec{\omega} \cdot \vec{r}} d\vec{\omega} \quad \underbrace{d\vec{\omega} = d\omega_1 d\omega_2}$$

2-D
Fourier
transform
pair.

2-D Fourier Transform of Derivatives

$$1) \quad F\left[\frac{\partial u}{\partial t}\right] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(\vec{r}, t) e^{i\vec{\omega} \cdot \vec{r}} d\vec{r} =$$

$$= \frac{\partial}{\partial t} \left[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\vec{r}, t) e^{i\vec{\omega} \cdot \vec{r}} d\vec{r} \right] = \frac{\partial}{\partial t} [U(\omega_1, \omega_2, t)]$$

$$= \frac{\partial}{\partial t} [F[u]]$$

$$2) \quad F\left[\frac{\partial u}{\partial x}\right] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x, y, t) e^{i(\omega_1 x + \omega_2 y)} dx dy$$

then,

$$F\left[\frac{\partial u}{\partial x}\right] = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,y,t) e^{i\omega_1 x} dx \right) e^{i\omega_2 y} dy$$

$\uparrow$
fix

$$= \frac{1}{(2\pi)} \int_{-\infty}^{\infty} -i\omega_1 V(\omega_1, \omega_2, t) e^{i\omega_2 y} dy = -i\omega_1 V(\omega_1, \omega_2, t)$$

$$= -i\omega_1 F[u].$$

Similarly,

$$F\left[\frac{\partial u}{\partial y}\right] = -i\omega_2 F[u].$$

$$F[\nabla^2 u] = F[u_{xx}] + F[u_{yy}] = -\omega_1^2 V(\omega_1, \omega_2, t) - \omega_2^2 V(\omega_1, \omega_2, t)$$

$$= -\omega^2 V(\omega_1, \omega_2, t).$$

$$= -\omega^2 F[u], \quad \|\vec{\omega}\|^2 = \omega_1^2 + \omega_2^2$$

Application to Heat Equ.

$$\begin{cases} u_t = k(u_{xx} + u_{yy}) = k \nabla^2 u \\ u(x, y, 0) = f(x, y) \end{cases}$$

The equ. is transformed into

$$\frac{dV}{dt} = -k \omega^2 V,$$

where

$$V = V(\omega_1, \omega_2, t) = F[u]$$

or

$$V(\vec{\omega}, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, t) e^{i\vec{\omega} \cdot \vec{r}} dx dy$$

$\omega_1 x + \omega_2 y$

Soln:

$$V(\vec{\omega}, t) = A(\vec{\omega}) e^{-k\omega^2 t}$$

Using I.C.

$$A(\vec{\omega}) = U(\vec{\omega}, 0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overset{f(x,y)}{U(x,y,0)} e^{i\vec{\omega} \cdot \vec{r}} dx dy$$

$$\therefore \boxed{U(\vec{\omega}, t) = F(\vec{\omega}) e^{-k\omega^2 t}} = F(\vec{\omega}) \quad (6.1)$$

Convolution thm in 2-D.

If $H(\vec{\omega}) = F(\vec{\omega}) G(\vec{\omega})$ and

$$F^{-1}[F(\vec{\omega})] = f(x,y) = f(\vec{r})$$

$$F^{-1}[G(\vec{\omega})] = g(x,y) = g(\vec{r})$$

then

$$\boxed{h(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{r}_0) g(\vec{r} - \vec{r}_0) d^2 r_0} \\ \text{Proof: - } = \text{same interchanging } f \leftrightarrow g. \quad (6.2)$$

$$h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\vec{\omega}) G(\vec{\omega}) e^{-i\vec{\omega} \cdot \vec{r}} d^2 \omega$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\vec{\omega}) \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{r}_0) e^{i\vec{\omega} \cdot \vec{r}_0} d^2 r_0 \right) e^{-i\vec{\omega} \cdot \vec{r}} d^2 \omega$$

changing order of integr.

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{r}_0) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\vec{\omega}) e^{-i\vec{\omega} \cdot (\vec{r} - \vec{r}_0)} d^2 \omega \right) d^2 r_0$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{r}_0) g(\vec{r} - \vec{r}_0) d^2 r_0 \quad \checkmark$$

Back to Soln. in freq. domain for heat equ.

In order to apply convolution thm,

We need the inverse ^{2-D} F.T. of $e^{-k\omega^2 t}$, let's call it $g(x,y)$

We already have the I.F.T. for $F(\vec{\omega}) \rightarrow f(\vec{r}) = f(x,y)$

$$g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\overbrace{k\omega_1^2 t - k\omega_2^2 t}^{-k\omega^2 t}} e^{-i\overbrace{\omega_1 x + \omega_2 y}^{\vec{\omega} \cdot \vec{r}}} d\omega_1 d\omega_2 =$$

$$\begin{aligned} & \stackrel{(*)}{=} \int_{-\infty}^{\infty} e^{-k\omega_1^2 t} e^{-i\omega_1 x} d\omega_1 \int_{-\infty}^{\infty} e^{-k\omega_2^2 t} e^{-i\omega_2 y} d\omega_2 = \\ & = \sqrt{\frac{\pi}{kt}} e^{-x^2/4kt} \sqrt{\frac{\pi}{kt}} e^{-y^2/4kt} = \frac{\pi}{kt} e^{-\frac{(x^2+y^2)}{4kt}} \end{aligned}$$

Therefore, applying convol. thm,

$$U(x,y,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_0, y_0) \frac{\pi}{kt} e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{4kt}} dx_0 dy_0$$

or

$$U(x,y,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_0, y_0) \left(\frac{1}{4\pi kt} e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{4kt}} \right) dx_0 dy_0$$

influence function
or
fund. soln.

$$\begin{aligned} & \stackrel{(*)}{\text{Inv. F.T. of}} e^{-k\omega^2 t} \rightarrow \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}} \\ & \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega \end{aligned}$$