Complex form of Fourier Series.

$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right]}$$

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,} \qquad \boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx.}$$

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \begin{aligned} \cos\theta &= \left(e^{i\theta} + e^{-i\theta}\right)/2 \\ \sin\theta &= \left(e^{i\theta} - e^{-i\theta}\right)/2i \end{aligned}$$

$$\theta_n = \frac{n\pi}{L}$$

$$\Rightarrow f(x) \sim a_0 + \sum_{n=1}^{\infty}\left\{a_n\left[\frac{e^{i\theta_n x} + e^{-i\theta_n x}}{2}\right] + b_n\left[\frac{e^{i\theta_n x} - e^{-i\theta_n x}}{2i}\right]\right\}$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(a_n - ib_n\right)e^{i\theta_n x} + \left(a_n + ib_n\right)e^{-i\theta_n x}\right]$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n - ib_n\right)e^{i\theta_n x} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n + ib_n\right)e^{-i\theta_n x} =$$

$m = -n$

$$= a_0 + \frac{1}{2}\sum_{m=-1}^{-\infty}\left(a_{-m} - ib_{-m}\right)e^{i\theta_m x} + \frac{1}{2}\sum_{n=1}^{\infty}(\ )$$

$$a_{-m} \overset{\text{even}}{=} a_m \,, \qquad b_{-m} \overset{\text{odd}}{=} -b_m \,, \qquad \theta_{-m} = -\theta_m$$

$$= a_0 + \frac{1}{2} \sum_{\substack{m=-\infty \\ \downarrow \\ n}}^{-1} (a_m + i b_m)\, e^{-\theta_m x} + \sum_{m=1}^{\infty} (a_m + i b_m)\, e^{-\theta_m x}$$

$$\Rightarrow \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n\, e^{-i\theta_n x} \,, \qquad C_0 = a_0 \,, \quad C_n = \frac{a_n + i b_n}{2} \,, \quad n \neq 0}$$

Similarly, if we change to keep : $e^{i\theta_n x}$.

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - i b_n)\, e^{i\theta_n x} + \frac{1}{2} \sum_{m=-1}^{-\infty} \underbrace{(a_{-m} + i b_{-m})}_{a_m - i b_m} e^{+i\theta_m x}$$

$$\Rightarrow \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} \bar{C}_n\, e^{i\theta_n x} \,, \qquad \bar{C}_0 = a_0 \,, \quad \bar{C}_n = \frac{a_n - i b_n}{2} \,, \ n \neq 0}$$

If $L = \pi$,

$$\Rightarrow \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n\, e^{-inx} \quad \text{or} \quad f(x) \sim \sum_{n=-\infty}^{\infty} \bar{C}_n\, e^{inx}}$$

Determination of the coefficients C_n and $\bar{C}_n$ in terms of the function $f(x)$.

(I) If $n \neq 0$, $n = \ldots -2, -1, 1, 2, \ldots$ $\quad C_n = \dfrac{a_n + ib_n}{2} =$

$$= \frac{1}{2L}\left[\int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x \, dx + i\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx\right] =$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi}{L}x}\, dx$$

for $n = 0$ $\quad C_0 = a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx = \dfrac{1}{2L}\int_{-L}^{L} f(x)\, e^{i0x}\, dx$

(II) If $n \neq 0$ $\quad \bar{C}_n = \dfrac{a_n - ib_n}{2} =$

$$= \frac{1}{2L}\left[\int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x \, dx - i\int_{-L}^{L} f(x) \sin \frac{n\pi}{L}x \, dx\right]$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\, dx.$$

Summarizing: Complex representation of Fourier Series.

(I)

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i\frac{n\pi}{L}x}} \qquad (3.1)$$

where

$$\boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi}{L}x}\,dx.} \qquad (3.2)$$

or

(II)

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{L}x}} \qquad (3.3)$$

where

$$\boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\,dx} \qquad (3.4)$$

Transition from a discrete set of eigenvalues to a Continuous set.

Substitution of (3.2) into (3.1) leads to

$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n\, e^{-i\frac{n\pi}{L}x} \Rightarrow$$

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^{L} f(\bar{x})\, e^{i\frac{n\pi}{L}\bar{x}}\, d\bar{x} \right] e^{-i\frac{n\pi}{L}x}} \quad (3.5)$$

Fourier series identity.

Calling $\omega = \frac{n\pi}{L}$ and $\Delta\omega = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$n = \cdots -2, -1, 0, 1, 2, \cdots$ $\Rightarrow \frac{1}{2L} = \frac{\Delta\omega}{2\pi}$

Subst. into (3.5)

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left[\int_{-L}^{L} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x} \right] e^{-i\omega x} \quad (3.6)$$

when $L \to \infty$, the distance between consecutive, $\omega = \sqrt{\lambda}$, $\Delta\omega \to 0$ $\Rightarrow$ that all real numbers are possible ω.

Therefore, when $L \to \infty$ equ. (3.6) can be written as

$$\boxed{f(x) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}\right] e^{-i\omega x}\, d\omega} \quad (3.7)$$

"Fourier integral identity."

We claim that the two improper integrals in the (3.7) expression converge if $f(x)$ is absolutely integrable: $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

For example, if $f(x)$ is piecewise smooth and $f(x) \xrightarrow[x \to \pm\infty]{} 0$ sufficiently fast, then $f(x)$ is absolutely integrable.

Now, we are ready for our definition:

<u>Def</u> - The function

$$\boxed{F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}}$$

is called the <u>Fourier Transform of $f(x)$</u>.

and equ. (3.7) can be written as

$$\boxed{f(x) \sim \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega}$$ Inverse Fourier transform of $F(\omega)$

Comparison

For $f(x)$ piecewise smooth on $[-L, L]$.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi}{L}x}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{\frac{in\pi}{L}x} dx.$$

Discrete Fourier Series of $f(x)$ on $[-L, L]$.

or

$$f(x) = \sum_{n=-\infty}^{\infty} \bar{c}_n e^{\frac{in\pi}{L}x}$$

$$\bar{c}_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi}{L}x} dx$$

For $f(x)$ absolutely integrable on $(-\infty, \infty)$.

$$f(x) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} dx$$

Inverse Fourier transform.

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega\bar{x}} d\bar{x}$$

Fourier transform of $f(x)$

$$f(x) = \int_{-\infty}^{\infty} C(\omega) e^{i\omega x} dx$$

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{-i\omega\bar{x}} d\bar{x}$$