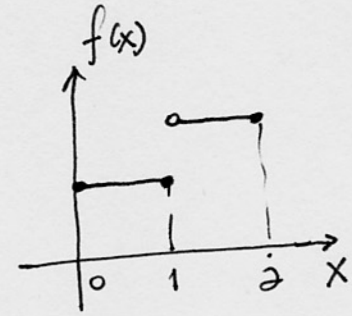
3.5 Term by Term Integration of Fourier Series.

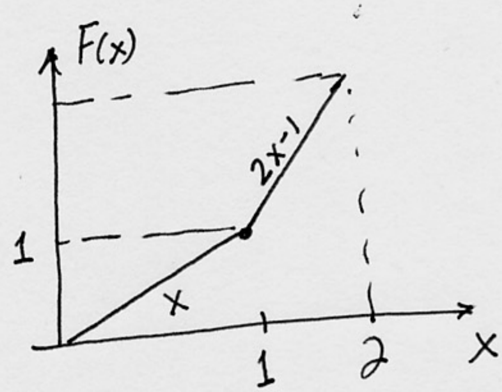
First, let's analyze this example

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2, & 1 < x \le 2. \end{cases}$$



$$F(x) = \int_0^x f(\bar{x})\,d\bar{x} = \begin{cases} \int_0^x 1\,d\bar{x} = x, & x \le 1 \\ \int_0^1 1\,d\bar{x} + \int_1^x 2\,d\bar{x} = 1 + 2(x-1) = 2x-1, & 1 < x \le 2 \end{cases}$$

or



$f(x)$ is piecewise continuous, but $F(x)$ is continuous on $[0,2]$.

This behavior is always the same for piecewise conts. functions.

In particular, Fourier series of piecewise smooth functions are always piecewise smooth

<u>Thm.</u>– If $f(x)$ p.w.s. on $[-L, L]$

then ① its F.S. can be integrated term by term

② The resulting series is convergent to $f(x)$ on $[-L,L]$.

<u>Rmk:</u> The new series (after integration) is continuous, but it may not be a F.S.

<u>Example.</u>– Consider the Fourier Sine Series of $f(x) \equiv 1$ on $[0,L]$

$$1 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{2}{L} \int_0^L \sin \frac{n\pi}{L} x \, dx = -\frac{2}{\cancel{L}} \frac{\cancel{L}}{n\pi} \cos\left(\frac{n\pi}{L} x\right)\Big|_0^L =$$

$$= -\frac{2}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ +\frac{4}{n\pi}, & n \text{ odd.} \end{cases}$$

or

$$\boxed{1 \sim \frac{4}{\pi}\left[\sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots\right]}$$

$0 \leq x \leq L$

Integrating term by term from 0 to x

$$x \sim \frac{4}{\pi}\left[-\frac{L}{\pi} \cos \frac{\pi \bar{x}}{L}\Big|_0^x - \frac{1}{3}\,\frac{L}{3\pi} \cos \frac{3\pi}{L} \bar{x}\Big|_0^x - \cdots\right]$$

or

$$x \sim \frac{4L}{\pi^2}\left[-\cos\frac{\pi}{L}x + 1 - \frac{1}{3^2}\left(\cos\frac{3\pi}{L}x - 1\right) - \cdots\right]$$

or

$$x \underset{\text{Thm}}{=} \frac{4L}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] - \frac{4L}{\pi^2}\left[\cos\frac{\pi}{L}x + \frac{1}{3^2}\cos\frac{3\pi}{L}x + \cdots\right]$$

$$0 \le x \le L \qquad (3.1)$$

Now, the F. cosine series of $f(x) = x$

$$x = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x, \qquad 0 \le x \le L$$

$$\boxed{a_0 = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}}$$

$$a_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi}{L}x\,dx = \frac{2L}{(n\pi)^2}(\cos n\pi - 1)$$

or

$$\boxed{a_n = \frac{2L}{(n\pi)^2}\left((-1)^n - 1\right) = \begin{cases} 0, & n \text{ even} \\ \dfrac{-4L}{(n\pi)^2}, & n \text{ odd} \end{cases}}$$

Therefore,

$$\frac{4L}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{L}{2}$$

Integrating (3.1) again from 0 to x leads to

$$\frac{x^2}{2} \sim \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{\sin 3\pi x/L}{3^3} + \frac{\sin 5\pi x/L}{5^3} + \cdots\right)$$

Not a Fourier Series because this term.