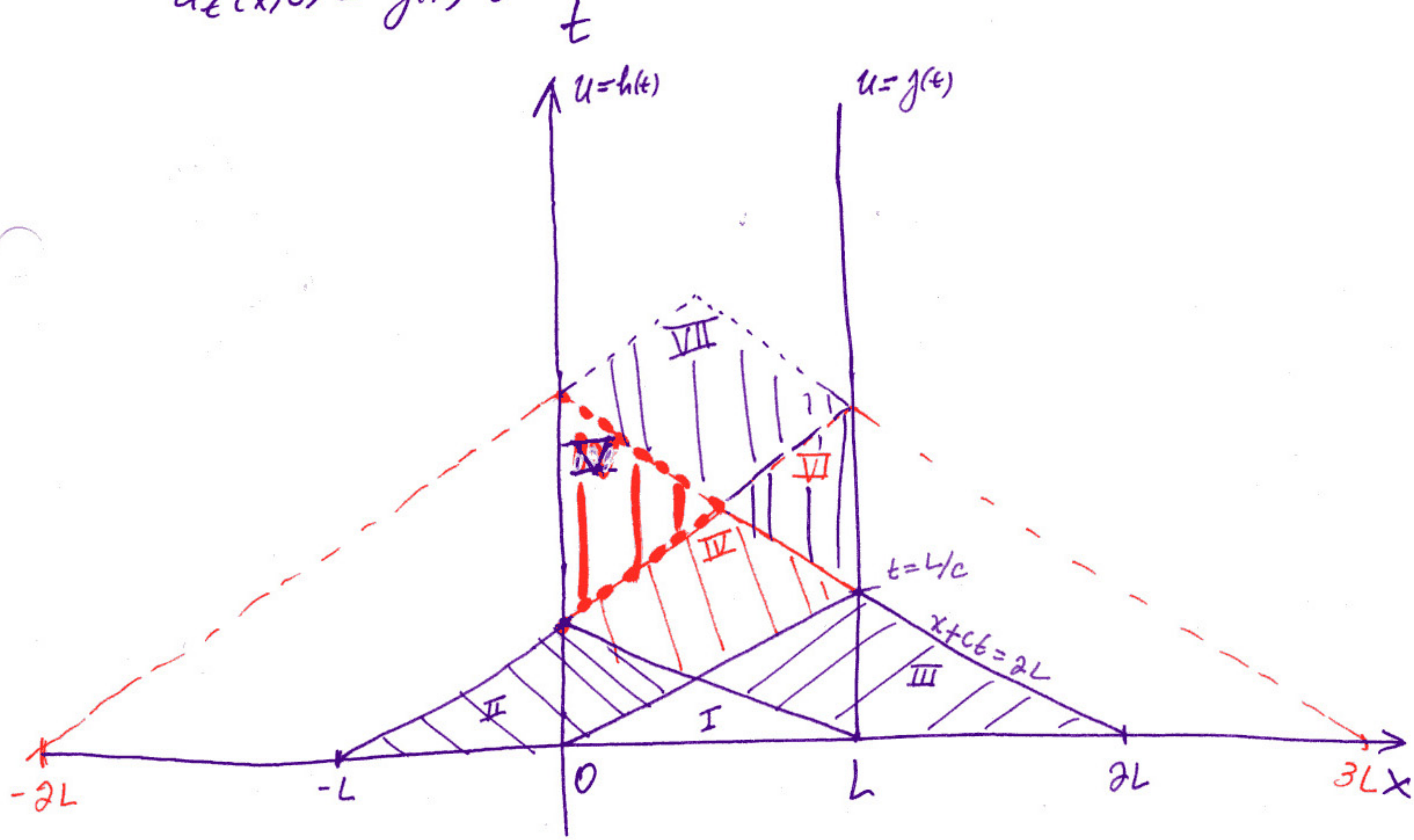
METHOD OF CHARACTERISTIC FOR THE WAVE EQUATION IN THE FINITE LINE.

$$u_{tt} = c^2 u_{xx}, \qquad 0<x<L, \; t>0$$

$$\left.\begin{aligned} u(0,t) &= h(t) \\ u(L,t) &= j(t) \end{aligned}\right\} \text{ BC's. DIRICHLET Type}$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{ IC's.}$$



General Soln:

$$u(x,t) = F(x-ct) + G(x+ct)$$

where
$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(z)\,dz, \quad 0<x<L$$
$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(z)\,dz, \quad 0<x<L$$

In region I: $0<x-ct<L, \quad 0<x+ct<L$

$$u(x,t) = F(x-ct) + G(x+ct)$$

or $$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

In region II: $-L<x-ct<0, \quad 0<x+ct<L$

$F(x-ct)$ is not defined

Using B.C at $x=0$

$$h(t) = u(0,t) = F(-ct) + G(ct)$$

$$\Rightarrow F(-ct) = h(t) - G(ct)$$

Calling $z=-ct \Rightarrow t = \frac{-z}{c}$ If $z \in [-L,0]$, $-z \in [0,L]$

$$\Rightarrow F_1(z) = h\left(\frac{-z}{c}\right) - G(-z)^{\text{ok}}$$

$$\Rightarrow \boxed{F_1(x-ct) = h\left(\frac{ct-x}{c}\right) - G(ct-x)}$$

In region II:

The Solution is given by

$$U(x,t) = F_1(x-ct) + G(x+ct) = h\left(t-\frac{x}{c}\right) - G(ct-x) + G(x+ct)$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz - \frac{1}{2c}\int_0^{ct-x} g(z)\,dz + h\left(t-\frac{x}{c}\right)$$

or

$$\boxed{U(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(z)\,dz + h(t-x/c).}$$

In region III: $0 < x-ct < L, \quad L < x+ct < 2L$

$G(x+ct)$ is not defined.

when $x+ct = 2L$

at $x=L \Rightarrow L+ct = 2L$

$\Rightarrow ct = L \Rightarrow t = L/c$

$\Downarrow$

Using BC at $x=L$

$$j(t) = U(L,t) = F(L-ct) + G_1(\overbrace{L+ct}^{>L})$$

$0 \le L-ct \le L$

$t=L/c \qquad t=0$

$$\Rightarrow G_1(L+ct) = j(t) - F(L-ct)$$

Calling $z = L+ct \Rightarrow t = \frac{z-L}{c} \Rightarrow ct = z-L$

$\Rightarrow L-ct = 2L-z$

$$G_1(z) = j\left(\frac{z-L}{c}\right) - F(2L-z) \quad \text{OK}$$

If $L<z<2L \Rightarrow -2L < -z < -L \Rightarrow 0 < 2L-z < L$

$$\Rightarrow \boxed{G_1(x+ct) = j\left(\frac{x+ct-L}{c}\right) - F(2L-x-ct)}$$

And the solution

$$U(x,t) = F(x-ct) + G_1(x+ct) = \frac{1}{2}\left[f(x-ct) - f(2L-x-ct)\right] + $$

$$- \frac{1}{2c}\int_0^{x-ct} g(z)\,dz + \frac{1}{2c}\int_0^{2L-x-ct} g(z)\,dz + j\left(t + \frac{x-L}{c}\right)$$

or In region 3:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) - f(2L-x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{2L-x-ct} g(z)\,dz + j\left(t+\frac{x-L}{c}\right)$$

In region IV: $-L < x-ct < 0$, $L < x+ct < 2L$

$$u(x,t) = F(x-ct) + G(x+ct) = F_1(x-ct) + G_1(x+ct)$$

$$= h\left(\frac{ct-x}{c}\right) - G(ct-x) + j\left(\frac{x+ct-L}{c}\right) - F(2L-x-ct).$$

– F_1 was obtained in terms of original G.

G_1 " " " " " " F.

In region V: $-2L < x-ct < -L$, $L < x+ct < 2L$

$$u(x,t) = \overset{?}{F_2}(x-ct) + \overset{\checkmark}{G_1}(x+ct)$$

$$h(t) = u(0,t) = F_2(-ct) + G_1(ct) \Rightarrow F_2(-ct) = h(t) - G_1(ct)$$

$$\Rightarrow F_2(z) = h(-z/c) - G_1(-z) \quad \text{OK}$$

If $-2L < z < -L \Rightarrow L < -z < 2L$

$$\Rightarrow \boxed{F_2(x-ct) = h\left(\frac{ct-x}{c}\right) - G_1(ct-x) = h(t-x/c) - j\left(\frac{x+ct-L}{c}\right) + F(2L+x-ct)}$$

In region VI: $\quad -L < x-ct < 0, \quad 2L < x+ct < 3L$

$$u(x,t) = \overset{\checkmark}{F_1}(x-ct) + \overset{?}{G_2}(x+ct)$$

B.C.

$$g(t) = u(L,t) = F_1(L-ct) + G_2(L+ct)$$

$$\Rightarrow G_2(L+ct) = g(t) - F_1(L-ct)$$

$z = L+ct$ $\qquad ct = z-L$, $\quad -ct = L-z$

$$G_2(z) = g\left(\frac{z-L}{c}\right) - F_1(2L-z)$$

on $-L < 2L-z < 0$

If $2L < z < 3L \Rightarrow -3L < -z < -2L \Rightarrow -L < 2L-z < 0$

$\therefore$

$$u(x,t) = F_1(x-ct) + G_2(x+ct) =$$

$$= h\left(\frac{ct-x}{c}\right) - G(ct-x) + G_2(x+ct) =$$

$$= h(t-x/c) - G(ct-x) + g\left(\frac{x+ct-L}{c}\right)$$

$$- F_1(2L-(x+ct))$$

$$= h(t-x/c) - G(ct-x) + g\left(\frac{x+ct-L}{c}\right)$$

$$- h\left(\frac{-2L+x+ct}{c}\right) - G(x+ct-2L)$$

In region VII: $\quad -2L < x-ct < -L, \quad 2L < x+ct < 3L$

$$\boxed{u(x,t) = \overset{\checkmark}{F_2}(x-ct) + \overset{\checkmark}{G_2}(x+ct)}$$

$$\begin{cases} u(x,0) = f(x) = \begin{cases} 1, & 4<x<5 \\ 0, & \text{otherwise} \end{cases} \\ u_t(x,0) = g(x) = 0 \end{cases}$$

$$u_{tt} = c^2 u_{xx}$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$