Semi-infinite string and reflections (III)

12.4.1 (More general)

$$\begin{cases} u_t = c^2 u_{xx}, & x>0,\ t>0 & (1) \\ u(x,0) = f(x), & x>0 & (2) \\ u_t(x,0) = g(x), & x>0 & (3) \\ u(0,t) = h(t), & t>0 & (4) \end{cases}$$

General soln. of (1): $\boxed{u(x,t) = F(x-ct) + G(x+ct)}$ (6.1)

Using IC's it is possible to determine F and G for $x>0$.

In fact,

$$f(x) = u(x,0) = F(x) + G(x), \quad x>0 \quad (5)$$

$$g(x) = u_t(x,0) = -cF'(x) + cG'(x), \quad x>0 \quad (6)$$

$$\Rightarrow f'(x) = F'(x) + G'(x)$$

$$\Rightarrow f'(x) + \frac{1}{c} g(x) = 2G'(x) \Rightarrow$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} + K$$

From (5) $F(x) = f(x) - G(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} - K.$

Summarizing,

$$\boxed{F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} - K, \quad x>0} \quad (5.2)$$

$$\boxed{G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} + K, \quad x>0} \quad (5.3)$$

Therefore, the Soln. of (1)-(4) is Completely Known when

(I) $x-ct>0$

$$u(x,t) = F(x-ct)+G(x+ct) = \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} \quad (7)$$

However, when

(II) $x-ct<0$

$F(x-ct)$ is not defined. **yet!** because $f(\underbrace{x-ct}_{<0})$ is not defined for negative argument.

To obtain the Soln. in this region, we use (5.1) and (4).

$$h(t) = u(0,t) = F(\underbrace{-ct}_{<0}) + G(\underbrace{ct}_{>0}) \quad (8)$$

$F(-ct)$ is not defined. However, from (8)

$$F(-ct) = h(t) - G(ct)$$

if $z=-ct$ then, $ct=-z$, and $t=-\frac{z}{c}$

Thus,

$$\boxed{F(z) = h\left(\underbrace{-\tfrac{z}{c}}_{>0}\right) - G(\underbrace{-z}_{>0}), \quad z<0} \quad (9) \Rightarrow \boxed{F(x-ct) = h\left(\tfrac{ct-x}{c}\right) - G(ct-x)}$$

Equ. (9) represents the def. of F for negative numbers

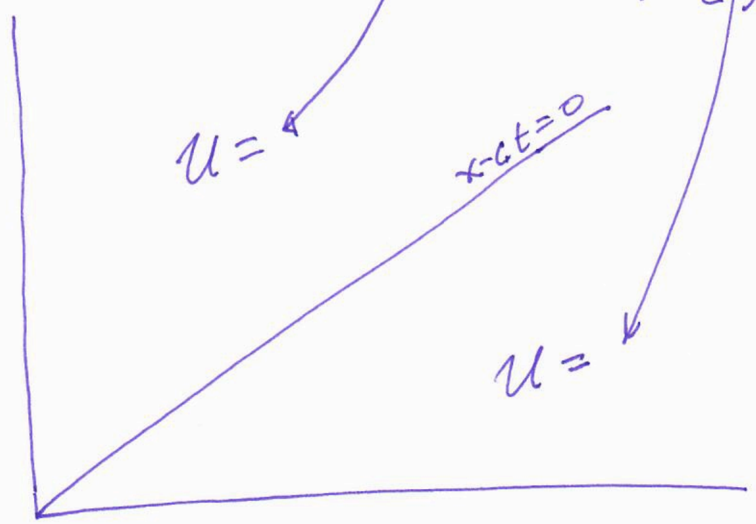
Therefore, the solution $u(x,t)$ when $x-ct<0$ can be written as

$$u(x,t) = F(x-ct)+G(x+ct) = h\left(\frac{ct-x}{c}\right) - \frac{1}{2}f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(\bar{x})\,d\bar{x} + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\bar{x})\,d\bar{x}$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x} + h\left(t-\frac{x}{c}\right)} \quad (10)$$

Therefore, Solution is given by

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}, & x-ct>0. \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x}, & x-ct<0 \\ \qquad + h\left(t-\frac{x}{c}\right) & \end{cases}$$



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12.4 Semi-Infinite String and Reflections.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t>0,\ x>0 \\ u(x,0) = \begin{cases} 1, & 4<x<5 \\ 0, & x<4 \text{ or } x>5 \text{ and } x>0. \end{cases} \\ u_t(x,0) = 0 \end{cases} \Bigg\} \text{IC's.}$$

$$u(0,t) = 0 \longrightarrow \text{B.C.}$$

Starting point:

General soln. : $u(x,t) = F(x-ct) + G(x+ct)$
for any (x,t) in $\{(x,t): x>0,\ t>0\}$.

General Case

Also, if I.C. are given by

$$\boxed{u(x,0) = f(x) \text{ and } u_t(x,0) = g(x),} \quad x>0$$

then,

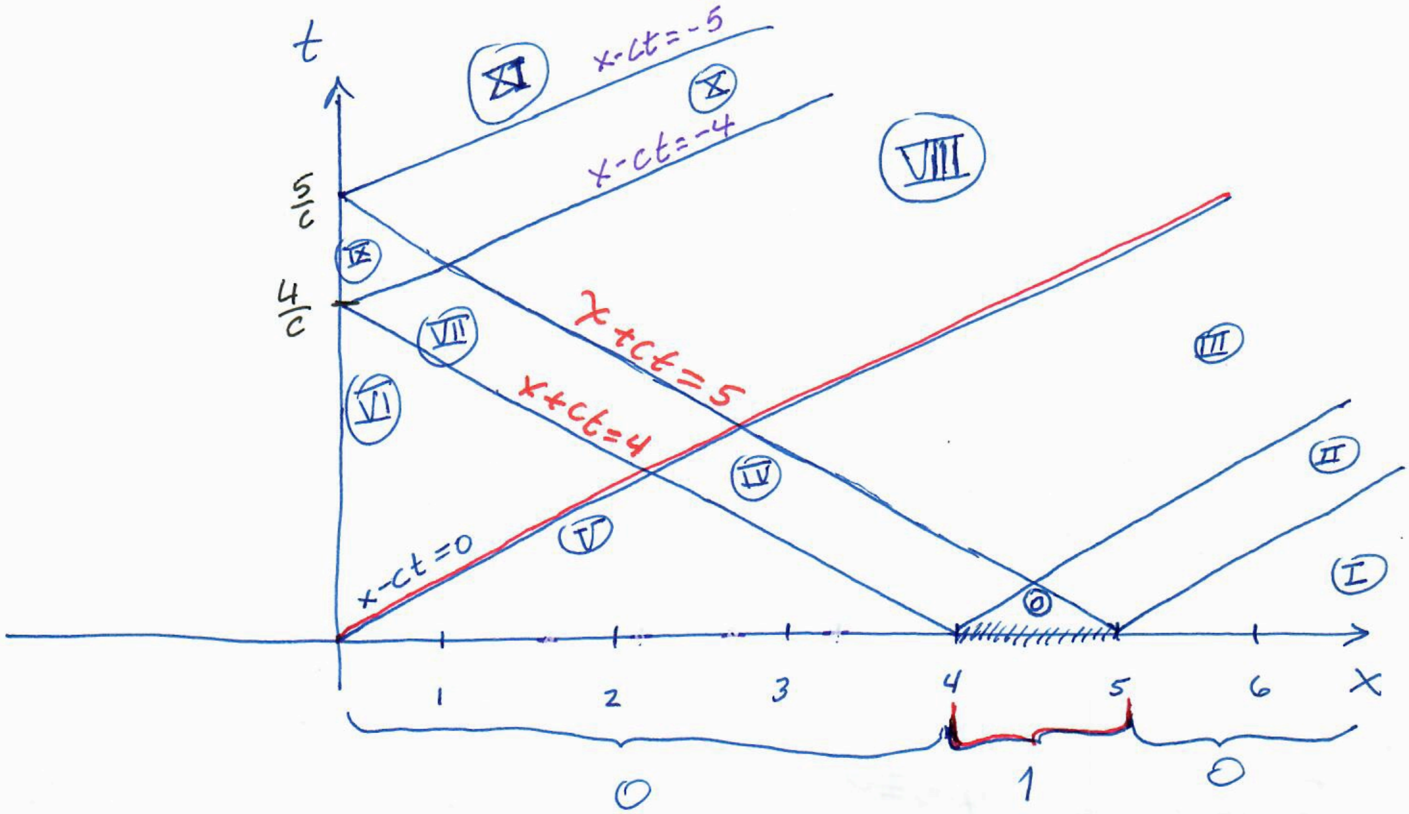
$$\boxed{F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} + k,} \quad x>0$$

$$\boxed{G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} - k,} \quad x>0$$

In our case,

$$F(x) = G(x) = \begin{cases} \frac{1}{2}, & 4<x<5 \\ 0, & x>5,\ x<4 \text{ and } x>0. \end{cases},$$

Let's Consider the xt-plane and the characteristics.



In regions: I - V   Same as in Infinite String.
No boundary effect below x-ct=0

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$ Turn

All Characteristics are located in the region $x-ct>0$ below the line $x-ct=0$.

At regions: I, V, and III $u(x,t) \equiv \frac{1}{2}[0+0] = 0$

At regions: IV and II, $u(x,t) \equiv \frac{1}{2}[1+0] = \frac{1}{2}$

At region 0: $u(x,t) \equiv \frac{1}{2}[1+1] = 1.$

In all regions: VI – XI

$\underline{x-ct<0}$ and $x+ct>0$.

and $u(x,t) = F(x-ct) + G(x+ct)$.

The problem is that $F(x)$ is only defined for $x>0$.

and $x-ct<0$

Thus, we need to find $F(x)$ for $x<0$

In order to do this, we use the B.C. at $x=0$.

Since $u(0,t)=0 \Rightarrow F(-ct)+G(ct)=0$

$$\Rightarrow F(\overset{<0}{-ct}) = -G(\overset{>0}{ct})$$

Calling $\boxed{y=-ct} \Rightarrow \boxed{t=-y/c}$

$$F(y) = -G(-y), \quad \text{for any } y<0.$$

Therefore, the general solution on this part of the domain is given by

$$u(x,t) = F(x-ct)+G(x+ct) = -G(\overset{>0}{\overline{ct-x}}) + G(\overset{>0}{\overline{x+ct}}), \quad \text{when } x-ct<0$$

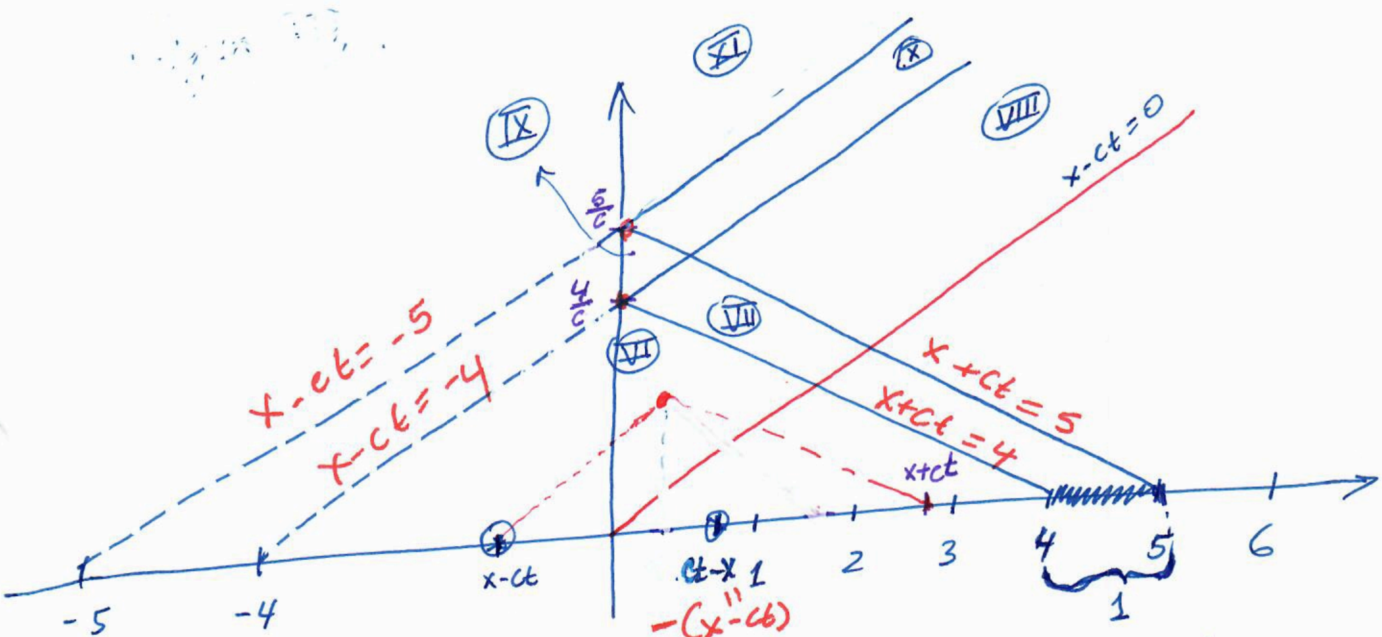
Using the definition of $G(x)$., $x>0$, for IC's: $u(x,0)=f(x)$, $u_t(x,0)=g(x)$.

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct)-f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x}.}, \quad \text{when } x-ct<0$$

In our particular case $g(x) \equiv 0$

and $u(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x) \right]$, $\quad x-ct<0$

Solution at the different regions in the plane



Inside Region VI: $-4 < x - ct < 0 \Rightarrow 0 < ct - x < 4$ and $0 < x+ct < 4$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[0-0\right] = 0$$

Inside Region VII: $-4 < x - ct < 0 \Rightarrow 0 < ct - x < 4$ and $4 < x+ct < 5$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[1-0\right] = \frac{1}{2}.$$

Inside Region IX: $-5 < x - ct < -4 \Rightarrow 4 < ct - x < 5$ and $4 < x+ct < 5$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\frac{1}{2}-\frac{1}{2}\right] = 0.$$

Inside region X: $-5 < x-ct < -4 \Rightarrow 4 < ct-x < 5$

and $x+ct > 5$

$$\Rightarrow u(x,t) = \frac{1}{2}[0-1] = -1/2.$$

Inside region VIII: $-4 < x-ct < 0 \Rightarrow 0 < ct-x < 4$

and $x+ct > 5$

$$\Rightarrow u(x,t) = \frac{1}{2}[0-0] = 0.$$

Inside region XI: $x-ct < -5 \Rightarrow ct-x > 5$

and $x+ct > 5$

$$\Rightarrow u(x,t) = \frac{1}{2}[0-0] = 0.$$

$$u(x,t) = \frac{1}{2}\left[f(x-ct) - f(ct-x)\right] \begin{cases} 0, & 0 < x+ct < 4, \quad \begin{matrix} -4 < x-ct < 0 \\ 0 < ct-x < 4 \end{matrix} \quad \text{VI}. \\ \frac{1}{2}, & 4 < x+ct < 5, \quad -4 < x-ct < 0 \quad \text{VII} \\ 0, & x+ct > 5, \quad -4 < x-ct < 0 \quad \text{VIII} \\ 0, & 4 < x+ct < 5, \quad -5 < x-ct < -4 \quad \text{IX} \\ -\frac{1}{2}, & x+ct > 5, \quad -5 < x-ct < -4 \quad \text{X} \\ 0, & x+ct > 5, \quad x-ct < -5 \quad \text{XI} \end{cases}$$

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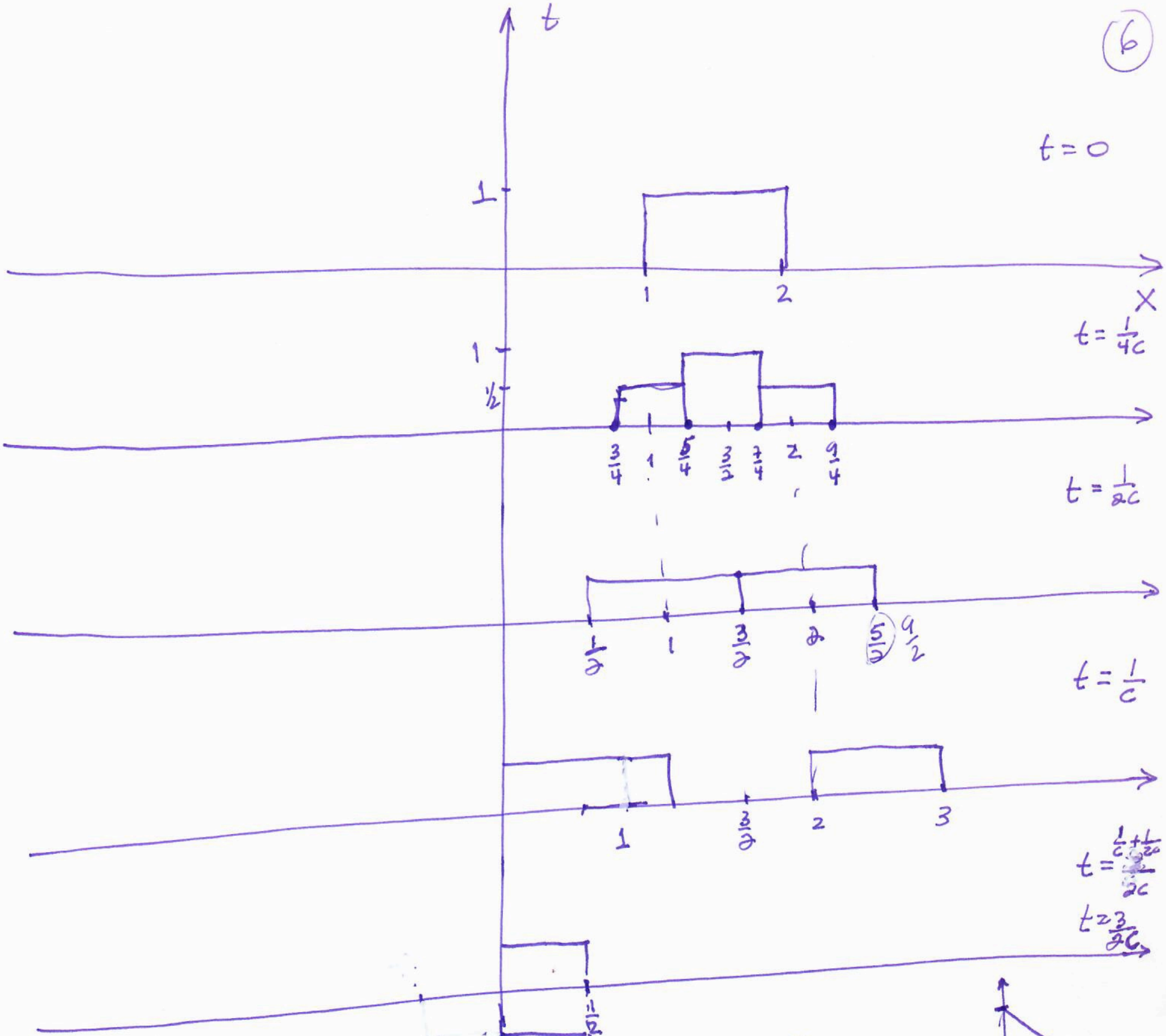
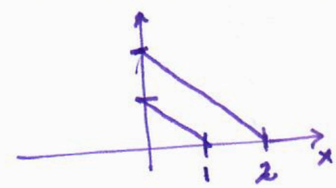




$x - ct = 1$

$t = \frac{1}{4c}$

$\Rightarrow x - \frac{1}{4} = 1$

$x = \frac{3}{4}$

$x + ct = 2$

$t = \frac{1}{4c}$

$\Rightarrow x = 2 - \frac{1}{4} = \frac{7}{4}$

At $t = \frac{3}{2c}$

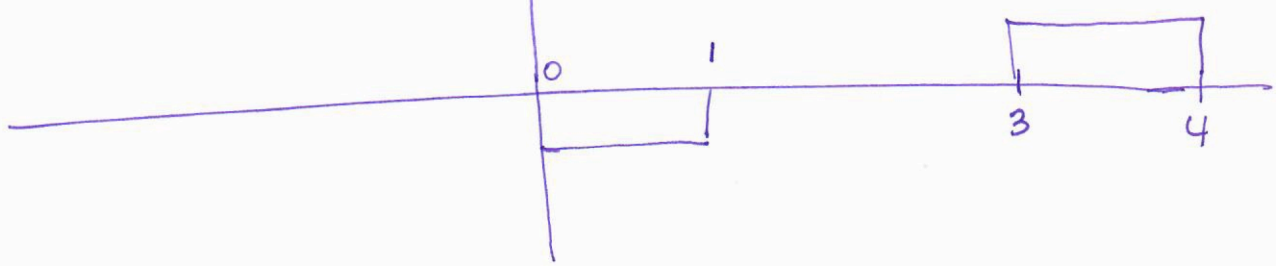
$x - ct = 1$

$x - c\left(\frac{3}{2c}\right) = 1 \Rightarrow x = \frac{5}{2}$

$x + \frac{3}{2} = 1 \Rightarrow x = -\frac{1}{2}$

$x + \frac{3}{2} = 2 \Rightarrow x = \frac{1}{2}$

$t = \frac{2}{c}$



Consider again the problem in the semi-infinite line

$$\text{Dirichlet IBVP} \begin{cases} u_t = c^2 u_{xx}, & 0<x<L,\ t>0 \quad (1) \\ u(0,t) = 0 & \quad (2) \\ u(x,0) = f(x), & u_t(x,0) = g(x) \quad (3) \end{cases}$$

Soln. is given by

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}, & x-ct>0 \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x}, & x-ct<0 \end{cases}} \quad (4)$$

Also, consider the problem in the infinite line

$$u_t = c^2 u_{xx}, \qquad -\infty<x<\infty,\ t>0 \quad (5)$$

$$u(x,0) = \hat{f}(x) = \begin{cases} f(x), & x>0, \\ -f(-x), & x<0 \end{cases}, \qquad u_t(x,0) = \hat{g}(x) = \begin{cases} g(x), & x>0 \\ -g(-x), & x<0 \end{cases} \quad (6)$$

odd extension of initial conditions

Thm.- The IBVPs (1)-(3) and (5)-(6) are equivalent over their common domains $0\le x<\infty$, $t>0$.

Proof.- The soln. of (5)-(6) using D'Alambert's formula is

$$u(x,t) = \frac{1}{2}\left[\hat{f}(x+ct) + \hat{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \hat{g}(\bar{x})\,d\bar{x} \Rightarrow$$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}, & x-ct>0. \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \hat{g}(\bar{x})\,d\bar{x} + \int_{0}^{x+ct} \hat{g}(\bar{x})\,d\bar{x}\right], & \begin{pmatrix} x-ct<0 \\ \text{and} \\ x+ct>0 \end{pmatrix} \end{cases}$$

Now, $\displaystyle\int_{x-ct}^{0} \hat{g}(\bar{x})\,d\bar{x} = \int_{x-ct}^{0} -g(-\bar{x})\,d\bar{x} \overset{u=-\bar{x}}{=} -\int_{u=ct-x}^{0} g(u)(-du) = \int_{ct-x}^{0} g(u)\,du$

and $\displaystyle\int_{0}^{x+ct} \hat{g}(\bar{x})\,d\bar{x} = \int_{0}^{x+ct} g(\bar{x})\,d\bar{x}$

therefore,

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x},} \quad \text{when} \begin{pmatrix} x-ct<0 \\ x+ct>0 \end{pmatrix}$$

Therefore, the solns of (1)-(3) and (5)-(6) are identical in their common domains. $0 \le x < \infty$, $t>0$.