7.5 Green's Formula. Self-adjoint operators and multidimensional EVP's.

For the 1-D Sturm-Liouville operator

$$L(u) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + qu$$

We obtained two important formulas:

a) Lagrange's identity:

$$\boxed{uLv - vLu = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]}$$

b) Green's formula:

$$\boxed{\int_a^b [uLv - vLu]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b}$$

For the 2-D or 3-D multidimensional EVP defined by the Laplace operator, $L \equiv \nabla^2$. There are similar formulas:

Divergence Theorem.

Concept of Region:- An open set containing all or some of the points forming its boundary.

Def.- A closed surface $\partial\Omega$ that consists of a finite number of smooth pieces joined together at the boundaries (curves defining boundaries) is called piecewise-smooth surface.

Def.- By a smooth surface S, we mean

$$S: \ \vec{r} = \vec{r}(u,v), \quad (u,v) \in D.$$

Such that $\vec{r}(u,v)$ is continuously differentiable and its unit normal vector $\hat{n}(\vec{x}_s)$ is continuous on S.

Theorem.- a) Ω is a bounded region.

b) $\partial\Omega$ is the closed piecewise smooth surface of Ω.

c) $F(\vec{x})$ is continuous on $\Omega \cup \partial\Omega$.

d) $F(\vec{x})$ is continuosly differentiable in Ω.

e) $\hat{n}(\vec{x}_s)$ is the unit outer normal vector to Ω at $\vec{x}_s$.

Then,

$$\iiint_{\Omega} \nabla \cdot \vec{F}(\vec{x})\, dv = \iint_{\partial\Omega} (\vec{F} \cdot \hat{n})(\vec{x}_s)\, ds.$$

1^2

$$\nabla \cdot (f\vec{G}) = \nabla \cdot \left(fg_1, fg_2, fg_3 \right) =$$

$$\frac{\partial (fg_1)}{\partial x} + \frac{\partial (fg_2)}{\partial y} + \frac{\partial (fg_3)}{\partial z} =$$

$$= f\frac{\partial G_1}{\partial x} + f\frac{\partial g_2}{\partial y} + f\frac{\partial g_3}{\partial z} + g_1\frac{\partial f}{\partial x} + g_2\frac{\partial f}{\partial y} + g_3\frac{\partial f}{\partial z}$$

$$= \underline{\underline{f\,\nabla \cdot \vec{G} + \vec{G} \cdot \nabla f}}.$$

Multidimensional Version of Lagrange's identity:

a) $$\boxed{u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v - v\nabla u)} \qquad (2.1)$$

b) Green's formula:

2-D: $$\iint_{\Omega} [u\nabla^2 v - v\nabla^2 u]\, dv = \oint_{\partial\Omega} [u\nabla v - v\nabla u]\cdot\hat{n}\, ds \qquad (2.2)$$

Ω is a Volume bounded by a sufficiently smooth surface $\partial\Omega$. Also, u and v are continuous together with their first derivatives inside $\Omega \cup \partial\Omega$ and they have conts. second derivatives in Ω.

3-D: $$\iiint_{\Omega} [u\nabla^2 v - v\nabla^2 u]\, dv = \iint_{\partial\Omega} [u\nabla v - v\nabla u]\cdot\hat{n}\, ds. \qquad (2.3)$$

Derivation of (2.1).: Work on this proof in class

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u\cdot\nabla v \Rightarrow u\nabla^2 v = \nabla\cdot(u\nabla v) - \cancel{\nabla u\cdot\nabla v}$$

$$\nabla\cdot(v\nabla u) = v\nabla^2 u + \nabla v\cdot\nabla u \Rightarrow v\nabla^2 u = \nabla\cdot(v\nabla u) - \cancel{\nabla v\cdot\nabla v}$$

$$u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v) - \nabla\cdot(v\nabla u)$$

or $$\boxed{u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v - v\nabla u)}.$$

Derivation of Green's Formula:

3D: Integrating (2.1)

$$\iiint_{\Omega} [u \nabla^2 v - v \nabla^2 u] \, d\vec{x} = \iiint_{\Omega} \nabla \cdot (u \nabla v - v \nabla u) \, d\vec{x} \tag{2.3}$$

$$\underset{\text{Divergence Thm}}{\overset{\text{Gauss or}}{=}} \iint_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \hat{n} \, ds$$

2D:

$$\iint_{\Omega} (u \nabla^2 v - v \nabla^2 u) \, d\vec{x} = \iint_{\Omega} \nabla \cdot (u \nabla v - v \nabla u) \tag{2.2}$$

$$\underset{2D}{\overset{\text{Diverg. thm}}{=}} \int_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \hat{n} \, dl$$

Notice the following:

i) If u and v and their derivatives are continuous on $\Omega \cup \partial\Omega$.

ii) If they have continuous 2^{nd} derivatives in Ω.

iii) If They satisfy boundary conditions of the regular Sturm-Liouville type:

$$\beta_1 w + \beta_2 \nabla w \cdot \hat{n}(\vec{x}_s) = 0, \qquad \vec{x}_s \in \partial\Omega. \quad \beta_1, \beta_2 \in \mathbb{R}.$$

Then,

$$\iint_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \hat{n}\, ds = 0$$

(Proof in the next page.)

Therefore, (2.3) and (2.2) reduce to

$$\iiint_{\Omega} (u \nabla^2 v - v \nabla^2 u)\, d\vec{x} = 0$$

$$\iint_{\Omega} (u \nabla^2 v - v \nabla^2 u)\, ds = 0.$$

We have said before, that if u and v satisfy

$$\beta_1 w + \beta_2 \nabla w . \hat{n} \, (\vec{x}_s) = 0, \qquad \vec{x}_s \in \partial\Omega.$$

$$w = u \text{ or } w = v.$$

$$\Rightarrow \iint_{\partial\Omega} (u \nabla v - v \nabla u) . \hat{n} \, ds = 0. \qquad (4.1)$$

Proof:- If ① $\beta_2 = 0$ Dirichlet BC's.
$\beta_1 \neq 0 \Rightarrow u(\vec{x}_s) = 0, \quad v(\vec{x}_s) = 0$

$$\Rightarrow \iint_{\partial\Omega} (\overset{0}{\cancel{u}} \nabla v - \overset{0}{\cancel{v}} \nabla u) . \hat{n} \, ds = 0$$

If ② $\beta_2 \neq 0$, $\beta_1 = 0$ $\Rightarrow$ Neumann cond. $\nabla u . \hat{n} \, (\vec{x}_s) = 0$, $\nabla v . \hat{n}(\vec{x}_s) = 0$

$$\Rightarrow \iint_{\partial\Omega} [u \overset{0}{\cancel{\nabla v . \hat{n}}} - v \overset{0}{\cancel{\nabla u . \hat{n}}}] \, ds = 0$$

If ③ $\beta_1 \neq 0$, $\beta_2 \neq 0$. $\Rightarrow \begin{cases} u = \dfrac{-\beta_2}{\beta_1} \nabla u . \hat{n} \\ v = -\dfrac{\beta_2}{\beta_1} \nabla v . \hat{n}. \end{cases}$

$$\Rightarrow \iint_{\Omega} [u \nabla v . \hat{n} - v \nabla u . \hat{n}] \, ds =$$

$$= \iint_{\Omega} \left[\left(-\frac{\beta_2}{\beta_1} \nabla u . \hat{n} \right) \nabla v . \hat{n} - \left(-\frac{\beta_2}{\beta_1} \nabla v . \hat{n} \right) \nabla u . \hat{n} \right] ds \overset{\text{clearly}}{=} 0$$

work also in BC's. where $\beta_2 \neq 0$ on part of $\partial\Omega$ and $\beta_2 = 0$ on another part.

Summarizing the differential operator "∇^2", applied to u and v sufficiently smooth on $\Omega \cup \partial\Omega$.

and u and v satisfying BCs of the regular Sturm-Liouville type,

Satisfies the equation:

$$\iiint_{\Omega} (u \nabla^2 v - v \nabla^2 u)\, d\vec{x} = 0.$$

So "∇^2" is a particular case of Self-adjoint operators that are defined in the next page.

Self-adjoint Operator

Consider a linear differential operator "L" acting on certain space of functions with given regularity.

$$L : \begin{pmatrix}\text{Space of}\\ \text{functions}\\ \text{Sufficiently}\\ \text{Smooth.}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Another}\\ \text{Space of}\\ \text{functions}\end{pmatrix}$$

$$u(\vec{x}) \longrightarrow Lu(\vec{x}).$$

$$v(\vec{x}) \longrightarrow Lv(\vec{x})$$

where the functions $u(\vec{x})$ and $v(\vec{x})$ are defined in a closed region $\underbrace{\Omega}_{\text{interior}} \cup \underbrace{\partial\Omega}_{\text{boundary}}$ and they satisfy certain boundary conditions (BCs) on $\partial\Omega$.

__Definition__. The operator L with its corresponding BCs is Self-adjoint if

$$\iiint_{\Omega} \left(u L[v] - v L[u]\right) dv = 0.$$

Compare "L" Self-adjoint with a symmetric matrix.

Regular Sturm-Liouville Eigenvalue Problem in 2D or 3D for Laplace Equation

$$\begin{cases} \nabla^2 u + \lambda u = 0 & \vec{x} \in \Omega. \quad (4.2) \\ \beta_1 u + \beta_2 \nabla u \cdot \hat{n}\,(\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega \quad (4.3) \end{cases}$$

We want to show that eigenfunctions corresponding to different eigenvalues are orthogonal with weight $\sigma = 1$.

It means if $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$ Satisfy

$$\begin{cases} \nabla^2 \phi_1 + \lambda_1 \phi_1 = 0 \\ \beta_1 \phi_1 + \beta_2 \nabla \phi_1 \cdot \hat{n} = 0 \end{cases} \qquad \begin{cases} \nabla^2 \phi_2 + \lambda_2 \phi_2 = 0 \\ \beta_1 \phi_2 + \beta_2 \nabla \phi_2 \cdot \hat{n} = 0 \end{cases}$$

then

$$\iiint_\Omega \phi_1(\vec{x})\, \phi_2(\vec{x})\, dv = 0. \quad \text{if } \lambda_1 \neq \lambda_2.$$

Orthogonality

Proof.– Idea: Use Green's Formula.

Replacing $u = \phi_1$ and $v = \phi_2$ into the Green's formula

$$\iiint_{\Omega} [\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1] \, dv = \iint_{\partial\Omega} [\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1] \cdot \hat{n} \, ds$$

Now, from previous discussion in pages 3-4, (equ. (4.1)) we know that for S-L regular Type BCs,

$$\iint_{\partial\Omega} [\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1] \cdot \hat{n} \, ds = 0.$$

then, $\displaystyle\iiint_{\Omega} [\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1] \, dv = 0$

or equivalently,

$$0 = \iiint_{\Omega} [\phi_1 (-\lambda_2 \phi_2) - \phi_2 (-\lambda_1 \phi_1)] \, dv$$

$$= (\lambda_1 - \lambda_2) \iiint_{\Omega} \phi_1(\vec{x}) \, \phi_2(\vec{x}) \, dv$$

Since $\lambda_1 \neq \lambda_2$, then

$$\iiint_{\Omega} \phi_1(\vec{x}) \, \phi_2(\vec{x}) \, dv = 0$$

7.6 Rayleigh Quotient and Laplace's Equation.

Consider the Sturm-Liouville multidimensional equation:

$$\nabla^2 \phi + \lambda \phi = 0, \qquad \vec{x} \in \Omega$$

multiplying by ϕ and integrating over Ω.

$$\iiint_\Omega \phi \nabla^2 \phi \, dv + \iiint_\Omega \lambda \phi^2 \, dv = 0$$

$$\Rightarrow \quad \boxed{\lambda = \frac{-\iiint_\Omega \phi \nabla^2 \phi \, dv}{\iiint_\Omega \phi^2 \, dv}} \qquad (5.1)$$

Integration by parts for multidimensional functions:

(1-D) $\quad \frac{d}{dx}(fg) = f \frac{dg}{dx} + \frac{df}{dx} g$

$\begin{pmatrix} \text{2-D} \\ \text{or} \\ \text{3-D} \end{pmatrix}$ If $f: \mathcal{D} \subset \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \to f(x,y,z)$ $\qquad \vec{G}: \mathcal{D} \subset \mathbb{R}^3 \to \mathbb{R}^3$, $(x,y,z) \to (g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$

$$\nabla \cdot (f\vec{G}) = f \nabla \cdot \vec{G} + \nabla f \cdot \vec{G}$$ (ask for a proof in class)

If $f = \phi$ and $\vec{G} = \nabla \phi$, then

$$\boxed{\nabla \cdot (\phi \nabla \phi) = \phi \nabla \cdot (\nabla \phi) + \nabla \phi \cdot \nabla \phi = \phi \nabla^2 \phi + \nabla \phi \cdot \nabla \phi.}$$

Or

$$\boxed{\nabla\cdot(\phi\nabla\phi) = \phi\,\nabla^2\phi + |\nabla\phi|^2} \qquad (6.1)$$

Solving for $\phi\nabla^2\phi$

$$\phi\,\nabla^2\phi = \nabla.(\phi\nabla\phi) - |\nabla\phi|^2.$$

Therefore, $$\iiint_\Omega \phi\nabla^2\phi\,dv \overset{\text{I.b.P.}}{=} \iiint_\Omega \nabla.(\phi\nabla\phi)\,dv - \iiint_\Omega |\nabla\phi|^2\,dv$$

$$\xlongequal{\text{Gauss Thm}} \iint_{\partial\Omega} \phi\nabla\phi\cdot\hat{n}\,ds - \iiint_\Omega |\nabla\phi|^2\,dv.$$

Substitution in (5.1).

$$\boxed{\lambda = \frac{-\iint_{\partial\Omega} \phi\nabla\phi.\hat{n}\,ds + \iiint_\Omega |\nabla\phi|^2\,dv}{\iiint_\Omega \phi^2(x)\,dv}} \qquad (6.1)$$

Application of Rayleigh Quotient

Theorem. The EVP

$$\begin{cases} \nabla^2\phi + \lambda\phi = 0, & \vec{x} \in \Omega \quad (7.1) \\ \text{with Dirichlet B.C.} \quad \phi(\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega. \quad (7.2) \end{cases}$$

has only positive eigenvalues.

Substituting the B.C. $\phi = 0$ in (6.1). results

$$\lambda = \frac{\iiint_\Omega |\nabla\phi|^2 dv}{\iiint_\Omega \phi^2(x) dx} \geq 0$$

If $\lambda = 0$, then $\nabla\phi = 0 \Rightarrow \frac{\partial\phi}{\partial x} = 0, \ \frac{\partial\phi}{\partial y} = 0$

$\Rightarrow \phi(\vec{x}) \equiv K$ (const), but $\phi(\vec{x}_s) = 0, \ \vec{x}_s \in \partial\Omega$

$\Rightarrow \quad \phi(\vec{x}) \equiv 0. \Rightarrow \phi(\vec{x})$ is not an eigenfunction and $\lambda = 0$ Cannot be an eigen Value.

Therefore, the BVP:

$$\begin{cases} \nabla^2\phi = 0, & \text{on } \Omega \\ \phi(\vec{x}_s) = 0, & \text{on } \partial\Omega \end{cases}$$

has only the trivial solution.

Corollary.

The BVP
$$\begin{cases} \nabla^2 w = 0, & \text{on } \Omega \quad (8.1) \\ w(\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega. \quad (8.2) \end{cases}$$
has only the trivial solution.

Proof.

The BVP (8.1) - (8.2) and (7.1) - (7.2) with $\lambda = 0$ are identical. Therefore only possible soln. for (8.1) - (8.2) is
$$w(x) \equiv 0.$$

Corollary. The BVP for $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Satisfying
$$\begin{cases} \nabla^2 u(\vec{x}) = f(\vec{x}), & \vec{x} \in \Omega \quad (8.3) \\ u(\vec{x}_s) = g(\vec{x}_s), & \vec{x} \in \partial\Omega \quad (8.4) \end{cases}$$
has a unique solution.

Proof. If $u_1(\vec{x})$ and $u_2(\vec{x})$ are two solutions of (8.3) - (8.4) then,

$w(\vec{x}) = u_1(\vec{x}) - u_2(\vec{x})$ is a solution of (8.1) - (8.2)

$$u_1(x) - u_2(\vec{x}) \equiv 0 \Rightarrow u_1(\vec{x}) = u_2(\vec{x}) \text{ in } \Omega.$$

Compare with similar result for invertible matrices $A_{n\times n}$ in linear algebra
$$A\vec{x} = \vec{b} \longrightarrow A\vec{x} = \vec{0} \longrightarrow A\vec{x} = \lambda\vec{x}.$$
Relationship btw them. ?

$$f(x, y) \sim \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{7.4.11}$$

5. **Orthogonality of eigenfunctions.** We will show that the multidimensional orthogonality of the eigenfunctions, as expressed by (7.4.5) for any two different eigenfunctions, can be used to determine the generalized Fourier coefficients in (7.4.4).[1] We will do this in exactly the way we did for one-dimensional Sturm–Liouville eigenfunction expansions. We simply multiply (7.4.4) by ϕ_{λ_i} and integrate over the entire region R:

$$\iint_R f\phi_{\lambda_i}\, dx\, dy = \sum_{\lambda} a_{\lambda} \iint_R \phi_{\lambda}\phi_{\lambda_i}\, dx\, dy. \tag{7.4.12}$$

Since the eigenfunctions are all orthogonal to each other (with weight 1 because $\nabla^2\phi + \lambda\phi = 0$), it follows that

$$\iint_R f\phi_{\lambda_i}\, dx\, dy = a_{\lambda_i} \iint_R \phi_{\lambda_i}^2\, dx\, dy, \tag{7.4.13}$$

or, equivalently,

$$\boxed{a_{\lambda_i} = \frac{\iint_R f\phi_{\lambda_i}\, dx\, dy}{\iint_R \phi_{\lambda_i}^2\, dx\, dy}.} \tag{7.4.14}$$

There is no difficulty in forming (7.4.14) from (7.4.13) since the denominator of (7.4.14) is necessarily positive.

For the special case that occurs for a rectangle with fixed zero boundary conditions, (7.4.7), the generalized Fourier coefficients a_{nm} are given by (7.4.14):

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dx\, dy}{\int_0^H \int_0^L \sin^2 \frac{n\pi x}{L} \sin^2 \frac{m\pi y}{H}\, dx\, dy}. \tag{7.4.15}$$

The integral in the denominator may be easily shown to equal $(L/2)(H/2)$ by calculating two one-dimensional integrals; in this way we rederive (7.3.30). In this case, (7.4.11), the generalized Fourier coefficient a_{nm} can be evaluated in two equivalent ways:

(a) Using one two-dimensional orthogonality formula for the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$

(b) Using two one-dimensional orthogonality formulas

[1] If there is more than one eigenfunction corresponding to the same eigenvalue, then we assume that the eigenfunctions have been made orthogonal (if necessary by the Gram–Schmidt process).

4b. **Convergence.** As with any Sturm–Liouville eigenvalue problem (see Section 5.10), a *finite* series of the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$ may be used to approximate a function $f(x, y)$. In particular, we could show that if we measure error in the mean-square sense,

$$E \equiv \iint_R \left(f - \sum_\lambda a_\lambda \phi_\lambda\right)^2 dx\, dy, \tag{7.4.16}$$

with weight function 1, then this mean-square error is minimized by the coefficients a_λ being chosen by (7.4.14), the generalized Fourier coefficients. It is known that the approximation improves as the number of terms increases. Furthermore, $E \to 0$ as all the eigenfunctions are included. We say that the series $\sum_\lambda a_\lambda\phi_\lambda$ **converges in the mean** to f.

EXERCISES 7.4

7.4.1. Consider the eigenvalue problem

$$\nabla^2\phi + \lambda\phi = 0$$

$$\frac{\partial\phi}{\partial x}(0, y) = 0, \quad \phi(x, 0) = 0$$

$$\frac{\partial\phi}{\partial x}(L, y) = 0, \quad \phi(x, H) = 0.$$

*(a) Show that there is a doubly infinite set of eigenvalues.

(b) If $L = H$, show that most eigenvalues have more than one eigenfunction.

(c) Show that the eigenfunctions are orthogonal in a two-dimensional sense using two one-dimensional orthogonality relations.

7.4.2. Without using the explicit solution of (7.4.7), show that $\lambda \geq 0$ from the Rayleigh quotient, (7.4.6).

7.4.3. If necessary, see Section 7.5:

(a) Derive that $\iint (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \oint (u\nabla v - v\nabla u)\cdot \hat{n}\, ds$.

(b) From part (a), derive (7.4.5).

7.4.4. Derive (7.4.6). If necessary, see Section 7.6. [*Hint*: Multiply (7.4.1) by ϕ and integrate.]

7.5 GREEN'S FORMULA, SELF-ADJOINT OPERATORS, AND MULTIDIMENSIONAL EIGENVALUE PROBLEMS

Introduction. In this section we prove some of the properties of the multidimensional eigenvalue problem:

$$\boxed{\nabla^2\phi + \lambda\phi = 0} \tag{7.5.1}$$