Interchanging $\sum$ and $\int$

$$\sum_{n=1}^{\infty} B_n \sinh\left(-\frac{n\pi H}{L}\right) \underbrace{\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx}_{=\begin{cases}0, & \text{if } n\neq m\\ \frac{L}{2}, & \text{if } n=m\end{cases}} =$$

$$= \int_0^L f_1(x)\sin\frac{m\pi}{L}x\,dx$$

$\Rightarrow$

$$B_n \frac{L}{2}\sinh\left(-\frac{n\pi H}{L}\right) = \int_0^L f_1(x)\sin\frac{n\pi}{L}x\,dx.$$

$\Rightarrow$

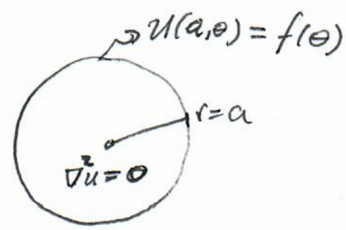
$$\boxed{B_n = \frac{2}{L\sinh\left(-\frac{n\pi H}{L}\right)}\int_0^1 f_1(x)\sin\left(\frac{n\pi}{L}x\right)dx.}$$

Laplace's Equation for a circular disk.

$$\begin{cases} \nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0. & 0<r<a,\ -\pi<\theta<\pi \\ u(a,\theta) = f(\theta), & -\pi\leq\theta\leq\pi \end{cases}$$

$\to$ physical boundary condition.

+ more conditions ?



We need conditions not only at $r=a$ but also $\underline{\underline{r=0}}$, $\underline{\underline{\theta=\pi}}$ and $\underline{\underline{\theta=-\pi}}$

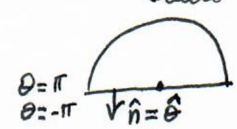
Extra physical conditions are

a) Temperature is finite at $r=0$.

i.e., $|u(0,\theta)| < \infty$.

b) Continuity of temperature and flux at $\underline{\underline{\theta=\pi}}$ Segment

Same as $\theta=-\pi$



$$\boxed{\begin{aligned} u(r,-\pi) &= u(r,\pi). \\ \frac{\partial u}{\partial \theta}(r,-\pi) &= \frac{\partial u}{\partial \theta}(r,\pi) \end{aligned}}$$

Why $\frac{\partial u}{\partial \theta}$ and not $\frac{\partial u}{\partial r}$.

$\vec{\phi}\cdot\hat{n}(r,\pi) = \vec{\phi}\cdot\hat{n}(r,-\pi)$

$-K_0 \nabla u\cdot\hat{n}(r,\pi) = -K_0 \nabla u\cdot\hat{n}(r,-\pi)$

or $\frac{\partial u}{\partial n}(r,\pi) = \frac{\partial u}{\partial n}(r,-\pi)$

$\hat{n}=\hat{\theta}$ $\quad \frac{\partial u}{\partial \theta}(r,\pi) = \frac{\partial u}{\partial \theta}(r,-\pi)$

Also, called <u>periodicity conditions</u>.

$$\begin{cases} \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 & (6.1) \\ u(a,\theta) = f(\theta). & (6.2) \\ |u(0,\theta)| < \infty & (6.3) \\ u(r,-\pi) = u(r,\pi). & (6.4) \\ \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) & (6.5) \end{cases}$$

All B.C's are homogeneous except (6.2).

Therefore, Separation of Variables applies.

$$\boxed{u(r,\theta) = \phi(\theta)\, G(r).}$$

Subst. leads to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0$$

Dividing by $\frac{1}{r^2}G(r)\phi(\theta)$.

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi(\theta)}\frac{d^2\phi}{d\theta^2} = \oplus\lambda$$

expect oscillations in θ.

$$\Rightarrow \quad \boxed{\frac{d^2\phi}{d\theta^2} + \lambda\,\phi(\theta) = 0} \qquad (7.1)$$

and

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda$$

or

$$\boxed{r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda\, G(r) = 0.} \qquad (7.2)$$

BC's (6.4) and (6.5) periodicity conds. implies

$$G(r)\,\phi(-\pi) = G(r)\,\phi(\pi) \xRightarrow{\text{for non-trivial solns.}} \phi(-\pi)=\phi(\pi).$$

$$G(r)\,\frac{d\phi}{d\theta}(-\pi) = G(r)\,\frac{d\phi}{d\theta}(\pi) \Rightarrow \phi'(-\pi)=\phi'(\pi).$$

Therefore, the eigenvalue problem is given by.

$$\begin{cases} \phi''(\theta) + \lambda\phi(\theta) = 0 \\ \phi(-\pi) = \phi(\pi) \\ \phi'(-\pi) = \phi'(\pi). \end{cases}$$

whose eigenvalues are $\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2,\ n=0,1,2\ldots$

and corresponding eigenfns. $\hat{\phi}_n(\theta) = \left\{ \begin{matrix} 1 \\ \cos(n\theta) \\ \sin(n\theta) \end{matrix} \right\}$ $n=1,2,\ldots$

To obtain the solns. for $G(r)$, we will use the condition: $|u(0,\theta)|<\infty \Rightarrow |\phi(\theta)\,G(0)|<\infty$
Since $|\phi(\theta)|<\infty$ this condition requires $|G(0)|<\infty$.

The solns. for (7.2) satisfy

$$\begin{cases} r^2\frac{d^2G_n}{dr^2} + r\frac{dG_n}{dr} - \overset{=n^2}{\lambda}\, G_n(r) = 0, \quad n=0,1,2\ldots & (8.1) \\ |G_n(0)|<\infty & (8.2) \end{cases}$$

Case 1) $\lambda = 0 \quad (n=0)$

Then $$\boxed{r^2 \frac{d^2 G_0}{dr^2} + r \frac{dG_0}{dr} = 0}$$

or $$\frac{d^2 G_0}{dr^2} + \frac{1}{r}\frac{dG_0}{dr} = 0 \Leftrightarrow \frac{1}{r}\frac{d}{dr}\left(r \frac{dG}{dr}\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left(r \frac{dG_0}{dr}\right) = 0 \Rightarrow r\frac{dG_0}{dr} = C_1$$

$$\Rightarrow \frac{dG_0}{dr} = \frac{C_1}{r} \Rightarrow \boxed{G_0(r) = C_1 \ln r + C_2}$$

Now, $|G_0(0)| < \infty \overset{C_1=0}{\Rightarrow} \boxed{G_0(r) = C_2}$

Case 2) $\lambda \neq 0, \quad n = 1, 2, \ldots$

$$\boxed{r^2 \frac{d^2 G_n}{dr^2} + r\frac{dG_n}{dr} - n^2 G_n(r) = 0}$$

Euler equation. Look for solns. as $G(r) = r^p$

$G_n'(r) = p\, r^{p-1}, \quad G_n''(r) = p(p-1)\, r^{p-2}$

Subst. leads to

$$p(p-1)\cancel{r^p} + p\cancel{r^p} - n^2 \cancel{r^p} = 0 \quad \text{or} \quad p(p-1) + p - n^2 = 0$$

$$p^2 - n^2 = 0 \Rightarrow p = \pm n.$$

Thus, $$\boxed{G_n(r) = C_1 r^n + C_2 r^{-n}.}$$

B.C. $\quad |G(0)| < \infty \Rightarrow \underline{C_2 = 0}$

As a consequence, $G_n(r) = C_1 r^n$.

Therefore, product solns:

$$A_0 (1), \quad \begin{matrix} A_n r^n \cos n\theta \\ B_n r^n \sin n\theta \end{matrix}$$

and infinite series soln. of BVP:

$$u(r,\theta) = \sum_{n=1}^{\infty} \left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \right] + A_0$$

Using I.C.

$$f(\theta) = u(a,\theta) = \sum_{n=1}^{\infty} \left[A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta) \right] + A_0$$

Using orthog.:

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta$$

$$B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta.$$