

Qualitative Properties of Laplace's Equation.

Consider the BVP:

$$\begin{cases} \nabla^2 u = 0, & \vec{x} \in \Omega \end{cases} \quad (1)$$

$$\begin{cases} u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega \end{cases} \quad (2)$$



where $h(\vec{x}_s)$ is conts. on $\partial\Omega$.

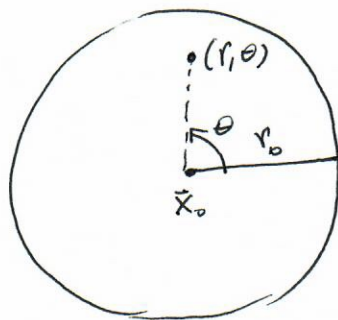
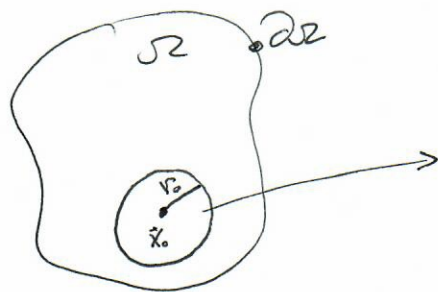
Concept of well-posedness

If BVP (1)-(2) has a unique solution that depends continuously on the boundary conditions, the BVP is said to be well-posed.

Thm (Mean Value theorem)

The value of $u(\vec{x}_0)$ soln. of (1)-(2) at any interior point $\vec{x}_0$ is the average of u along any circle of radius r_0 centered at $\vec{x}_0$.

Proof. For any circular disk ^{region bounded by a circle} $D(\vec{x}_0, r_0)$ centered at $\vec{x}_0$ of radius r_0 contained in Ω , we can define a polar coordinate system with center at $\vec{x}_0$.



$$r \equiv \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$\tan \theta = \frac{y-y_0}{x-x_0}$$

From our previous result for a circular disk,

$$U(r, \theta) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\theta) + B_n \sin(n\theta)] r^n, \quad \text{in } D(\tilde{x}_0, r_0)$$

In particular, at $r=0$

$$U(\tilde{x}_0) = U(0, \theta) = A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(r_0, \theta) d\theta = \text{Average of } U(r, \theta) \text{ along circle of radius } r_0.$$

Thm (Max pple)

The maximum and minimum of u soln. of (1)-(2) occur on the boundary $\partial\Omega$.

Proof - There are two cases:

i) The soln. u is constant everywhere including the boundaries.

In this case, the max. pple is trivially true.

ii) The soln u is not constant everywhere

If in this case, we assume that the max value of u occurs in the interior of Ω at a point $\vec{x} = \vec{x}_0$.

Let's construct an ^{arbitrary} circle $B_r(\vec{x}_0) \subset \Omega$. Then, according to the mean value theorem,

$$u(\vec{x}_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underset{u(\vec{x}_0)}{u(r_0, \theta)} d\theta, \quad \vec{x}_0 \in B_r(\vec{x}_0)$$

That means that it should be at least one point on $B_r(\vec{x}_0)$

where $u(\vec{x}_0) < u(\vec{x}_c)$ contradicting the hypothesis.

If not $u(\vec{x}_c) = u(\vec{x}_0)$, for all $\vec{x}_c \in B_r(\vec{x}_0)$,
for all radius r such that $B_r(\vec{x}_0) \subset \Omega$.

This leads to a constant solution or case (i), contradicting that u is not constant.

Thm. - The solution of BVP (1)-(2) satisfies:

- i) If its solution exists then it's unique.
- ii) " " " " " it depends continuously on the boundary cond.

Proof. - i) Assume $u_1(\vec{x})$ and $u_2(\vec{x})$ are two solutions of (1) satisfying (2), then

$$W(x) \equiv U_1(x) - U_2(x)$$

Satisfies (1) and $W(\vec{x}_s) = U_1(\vec{x}_s) - U_2(\vec{x}_s) = h(\vec{x}_s) - h(\vec{x}_s) \equiv 0$

$$\text{So } \begin{cases} \nabla^2 W = 0, & \vec{x} \in \Omega \\ W(\vec{x}_s) = 0, & \vec{x} \in \partial\Omega \end{cases} \quad \text{for all } \vec{x}_s \in \partial\Omega.$$

Using max pple

$$0 = \min_{\vec{x}_s \in \partial\Omega} (W(\vec{x}_s)) \leq W(x) \leq \max_{\vec{x}_s \in \partial\Omega} (W(\vec{x}_s)) = 0$$

$$\Rightarrow W(\vec{x}) \equiv 0, \quad \vec{x} \in \Omega \cup \partial\Omega \Rightarrow \boxed{U_1(\vec{x}) = U_2(\vec{x}), \text{ for all } \vec{x} \in \Omega \cup \partial\Omega.}$$

(ii) Assume $U(x)$ is a soln. of (1)-(2),

and $w(x)$ is a soln. of (1) with B.C. $U(\vec{x}_s) = g(\vec{x}_s)$
 $\vec{x}_s \in \partial\Omega.$

and $|h(\vec{x}_s) - g(\vec{x}_s)| < \varepsilon, \quad \varepsilon > 0.$

then $\nabla^2(U-w) = \nabla^2 U - \nabla^2 w = 0 - 0 = 0$, and $(U-w)(\vec{x}_s) = h(\vec{x}_s) - g(\vec{x}_s)$ on $\partial\Omega$

$$-\varepsilon < \min_{\vec{x}_s \in \partial\Omega} (h(\vec{x}_s) - g(\vec{x}_s)) \leq U(\vec{x}) - w(\vec{x}) \leq \max_{\vec{x}_s \in \partial\Omega} (h(\vec{x}_s) - g(\vec{x}_s)) < \varepsilon$$

$$\Rightarrow |U(\vec{x}) - w(\vec{x})| < \varepsilon, \quad \vec{x} \in \Omega \cup \partial\Omega.$$

Remark: $|h(\vec{x}_s) - g(\vec{x}_s)| < \varepsilon \Leftrightarrow -\varepsilon < h(\vec{x}_s) - g(\vec{x}_s) < \varepsilon$

In particular $\min (h(\vec{x}_s) - g(\vec{x}_s)) > -\varepsilon$
 $\max (h(\vec{x}_s) - g(\vec{x}_s)) < \varepsilon.$

2.5 Solvability condition

Problem 1.47(c) with $\beta=1$, $L=1$

$$\begin{cases} u_t = u_{xx} + (x-1) \\ u(x,0) = f(x) \\ u_x(0,t) = 0, \quad u_x(1,t) = 0 \end{cases}$$

Equilibrium problem:

$$\begin{cases} \hat{u}'' = -x+1 \\ \hat{u}'(0) = 0, \quad \hat{u}'(1) = 0 \end{cases}$$

$$\Rightarrow \hat{u}'(x) = -\frac{x^2}{2} + x + C_1$$

Using B.C. $\hat{u}'(0) = 0 \Rightarrow \boxed{C_1 = 0}$

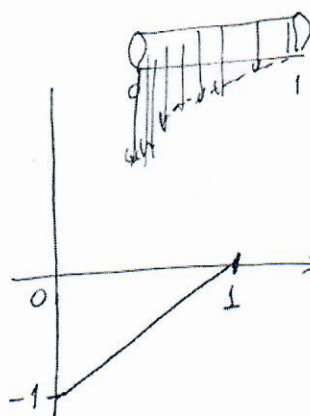
$$\hat{u}'(1) = 0 \Rightarrow -\frac{1}{2} + 1 = 0 \text{ Contradiction!}$$

There is not an equilibrium.

We also notice that

Using B.C.s. Neumann $\int_0^1 \hat{u}''(x) dx = \hat{u}'(x) \Big|_0^1 = \hat{u}'(1) - \hat{u}'(0) = 0$

and also using the Poisson equ. $\int_0^1 \hat{u}''(x) dx = \int_0^1 (-x+1) dx = -\frac{x^2}{2} + x \Big|_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$



Therefore, for the IBVP:

$$\begin{cases} u_t = K u_{xx} + Q(x) \\ u(x,0) = f(x) \\ u_x(0,t) = A, \quad u_x(L,t) = B \end{cases}$$

An equilibrium or solution for the BVP exists.

$$\begin{cases} \hat{u}''(x) = -\frac{Q(x)}{K} \\ \hat{u}'(0) = 0, \quad \hat{u}'(L) = 0 \end{cases}$$

$$\text{if } \int_0^L \hat{u}''(x) dx = \int_0^L -\frac{Q(x)}{K} dx$$

$$\Rightarrow \hat{u}'(L) - \hat{u}'(0) = -\int_0^L \frac{Q(x)}{K} dx$$

$$\text{or } \boxed{B - A = -\int_0^L \frac{Q(x)}{K} dx.}$$

This condition is usually called
Solvability or compatibility condition.

(3)

In 2-D or 3-D, the equilibrium problem
is given by for a Neumann problem

$$\begin{cases} \nabla^2 u = -\frac{Q(\vec{x})}{K}, & \vec{x} \in \Omega \\ \frac{\partial u}{\partial n}(\vec{x}_s) = f(\vec{x}_s), & \vec{x}_s \in \partial\Omega. \end{cases}$$

Therefore,
$$\iiint_{\Omega} \nabla^2 u(\vec{x}) dV = - \iiint_{\Omega} \frac{Q(\vec{x})}{K} dV$$

But also,
$$\iiint_{\Omega} \nabla^2 u(\vec{x}) dV \stackrel{\text{Gauss thm}}{=} \iint_{\partial\Omega} \nabla u \cdot \hat{n} ds$$

Then, a necessary cond. for an ^{equilibrium} solution to exist is

that
$$\iint_{\partial\Omega} (\nabla u \cdot \hat{n})(\vec{x}_s) ds = - \iiint_{\Omega} \frac{Q(\vec{x})}{K} dV$$

Compatibility condition.

flux across the surface $\partial\Omega$ = Sources or Sink of energy inside Ω .

In the case of a Laplace equation a ^{Neumann} BVP as

$$\begin{cases} \nabla^2 u = 0, & \vec{x} \in \Omega \\ \frac{\partial u}{\partial n}(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega \end{cases}$$

a necessary condition for a solution to exist is that

$$\begin{aligned} 0 &= \iiint_{\Omega} \nabla^2 u(\vec{x}) d\vec{v} \stackrel{\text{Gauss Thm}}{=} \iint_{\partial\Omega} \frac{\partial u}{\partial n}(\vec{x}_s) ds = \\ &= \iint_{\partial\Omega} h(\vec{x}_s) ds \end{aligned}$$

It means

$$\iint_{\partial\Omega} \nabla u \cdot \hat{n} ds = \iint_{\partial\Omega} h(\vec{x}_s) ds = 0.$$

Net flux
out of Ω
through $\partial\Omega$
is zero.

Solvability Condition

$$\# \quad 1.4.7 \quad \begin{cases} u''(x) = 0 \\ u'(0) = 1 \\ u'(L) = \beta \end{cases}$$

$$0 = \int_0^L 0 dx = \int_0^L u''(x) dx = u'(L) - u'(0) = \beta - 1 \Rightarrow \boxed{\beta = 1}$$

for a soln. to exist. otherwise there is no equilibrium.

$$\begin{cases} \nabla^2 u = 0 \\ \frac{\partial u}{\partial n} = f(\vec{x}) \end{cases}$$

$$0 = \iint_S \nabla^2 u(\vec{x}) ds = \int_{\partial S} (\nabla u) \cdot \hat{n} dl = \int_{\partial S} \frac{\partial u}{\partial n} dl = \int_{\partial S} f(\vec{x}) dl$$

So compatibility cond.

$$\Rightarrow \int_{\partial S} f(\vec{x}) dl = 0.$$

Before 2.5.4.

Solution for Laplace's equation inside a circular disk.

$$\begin{cases} \nabla_{r,\theta}^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 & (1) \end{cases}$$

$$\begin{cases} u(a, \theta) = f(\theta) & (2) \end{cases}$$

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \frac{r^n}{a^n} + \sum_{n=1}^{\infty} B_n \sin(n\theta) \frac{r^n}{a^n} \quad (3)$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta, \quad A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta \quad (4)$$

$$B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta \quad (5)$$

Subst. (4) - (5) into (3).

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\bar{\theta}) d\bar{\theta} + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(\bar{\theta}) \cos(n\bar{\theta}) d\bar{\theta} \right) \cos(n\theta) \frac{r^n}{a^n}$$

$$+ \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} f(\bar{\theta}) \sin(n\bar{\theta}) d\bar{\theta} \right) \sin(n\theta) \frac{r^n}{a^n}$$

2.5.4.

$$U(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\bar{\theta}) d\bar{\theta} + \sum_{n=1}^{\infty} \frac{1}{\pi} \left[\left(\int_{-\pi}^{\pi} f(\bar{\theta}) \cos n\bar{\theta} d\bar{\theta} \right) \frac{r^n}{a^n} \cos n\theta + \right. \\ \left. + \sum_{n=1}^{\infty} \frac{1}{\pi} \left[\int_{-\pi}^{\pi} f(\bar{\theta}) \sin n\bar{\theta} d\bar{\theta} \right] \frac{r^n}{a^n} \sin n\theta = \right.$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} f(\bar{\theta}) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n [\cos n\bar{\theta} \cos n\theta + \sin n\bar{\theta} \sin n\theta] \right] d\bar{\theta} \right\}$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} f(\bar{\theta}) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \cos(n\theta - n\bar{\theta}) \right] d\bar{\theta} \right\}$$

$$\therefore U(r, \theta) = \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} f(\bar{\theta}) \left[\frac{1}{2} + \sum_{n=0}^{\infty} \left(\frac{r}{a} \right)^n \cos(n\theta - n\bar{\theta}) \right] d\bar{\theta} \right\} \quad (1.1)$$

Using $\cos z = \operatorname{Re} [e^{iz}]$ sum the resulting geometric series to obtain Poisson's integral formula.

$$U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \frac{a^2 - r^2}{r^2 + a^2 - 2ar \cos(\theta - \theta_0)} d\theta_0 \quad (1.2)$$

Geometric Series: If $r < 1$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

The term (general) inside the Σ .

$$\begin{aligned} \left(\frac{r}{a}\right)^n \cos(n\theta - n\bar{\theta}) &= \left(\frac{r}{a}\right)^n \operatorname{Re}[e^{in(\theta - \bar{\theta})}] \\ &= \operatorname{Re}\left[\left(\frac{r}{a}\right)^n e^{in(\theta - \bar{\theta})}\right] = \operatorname{Re}\left[\left(\frac{r}{a} e^{i(\theta - \bar{\theta})}\right)^n\right] \end{aligned}$$

Subst. in the Σ .

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta - n\bar{\theta}) &= \sum_{n=0}^{\infty} \operatorname{Re}\left[\left(\frac{r}{a} e^{i(\theta - \bar{\theta})}\right)^n\right] \\ &= \operatorname{Re}\left(\sum_{n=0}^{\infty} \left(\frac{r}{a} e^{i(\theta - \bar{\theta})}\right)^n\right) = \operatorname{Re}\left(\frac{1}{1 - \frac{r}{a} e^{i(\theta - \bar{\theta})}}\right) \\ &\quad \left| \frac{r}{a} e^{i(\theta - \bar{\theta})} \right| = \frac{r}{a} < 1 \\ &= \operatorname{Re}\left[\frac{1}{1 - \frac{r}{a} \cos(\theta - \bar{\theta}) - i \frac{r}{a} \sin(\theta - \bar{\theta})}\right] \\ &= \operatorname{Re}\left[\frac{1 - \frac{r}{a} \cos(\theta - \bar{\theta}) + i \frac{r}{a} \sin(\theta - \bar{\theta})}{\left(1 - \frac{r}{a} \cos(\theta - \bar{\theta}) - i \frac{r}{a} \sin(\theta - \bar{\theta})\right) \left(1 - \frac{r}{a} \cos(\theta - \bar{\theta}) + i \frac{r}{a} \sin(\theta - \bar{\theta})\right)}\right] \end{aligned}$$