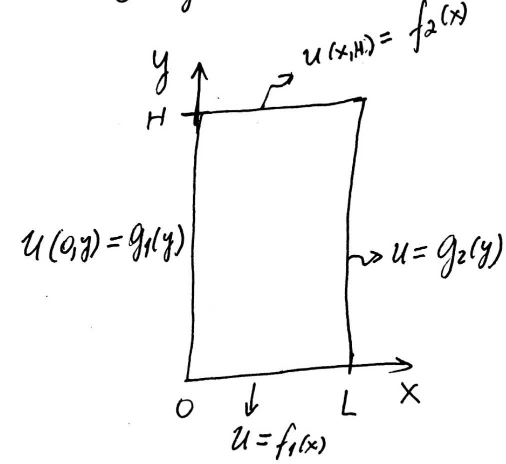
2.5.1 LAPLACE'S EQUATION INSIDE A RECTANGLE.

$$\nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0.$$

$$0 \le x \le L,$$
$$0 \le y \le H.$$

Boundary Conditions:

$$u(0,y) = g_1(y)$$
$$u(L,y) = g_2(y)$$
$$u(x,0) = f_1(x)$$
$$u(x,H) = f_2(x)$$

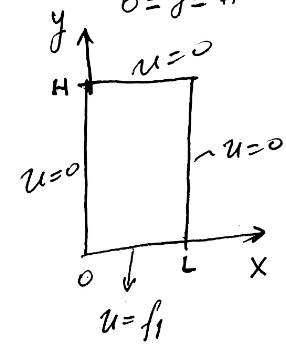


Non-homogeneous problem. Method of Separation Of Variables can't be applied.
However, linearity allows a decomposition of the soln. as

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Where each $u_i(x,y)$ has only one non-homogeneous B.C. For example, consider

$$\begin{cases} \nabla^2 u_1 = (u_1)_{xx} + (u_1)_{yy} = 0, & 0 \le x \le L \\ & 0 \le y \le H \\ \text{B.C.} \quad u_1(x,0) = f_1(x) & \\ \phantom{\text{B.C.}} \quad u_1(x,H) = 0 & \\ \phantom{\text{B.C.}} \quad u_1(0,y) = 0 & \\ \phantom{\text{B.C.}} \quad u_1(L,y) = 0 & \end{cases}$$



Sep. of Variables:

$$u_1(x,y) = h(y)\,\phi(x)$$

$$\Rightarrow \quad \phi''(x)\,h(y) + h''(y)\,\phi(x) = 0 \qquad \text{dividing by } \phi(x)h(y).$$

$$\Rightarrow \quad \frac{\phi''(x)}{\phi(x)} + \frac{h''(y)}{h(y)} = 0$$

$$\Rightarrow \quad \frac{\phi''(x)}{\phi(x)} = -\frac{h''(y)}{h(y)} = -\lambda$$

$$\Rightarrow \quad \boxed{\phi''(x) + \lambda\,\phi(x) = 0} \quad \text{and} \quad \boxed{h''(y) - \lambda\,h(y) = 0}$$

$$u_1(0,y) = 0 \Rightarrow \boxed{\phi(0) = 0}$$

$$u_1(L,y) = 0 \Rightarrow \boxed{\phi(L) = 0}$$

Eigenvalue problem

$$u_1(x,H) = 0 \Rightarrow \boxed{h(H) = 0}$$

We know from previous heat conduction IBVP that

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \phi_n(x) = \sin\left(\frac{n\pi}{L}x\right). \quad n = 1,2,\ldots}$$

Also $h''(y) - \lambda h(y) = 0$

$\Rightarrow \quad r^2 - \lambda = 0 \quad \Rightarrow \quad r = \pm\sqrt{\lambda} = \pm\sqrt{\left(\frac{n\pi}{L}\right)^2} = \pm\frac{n\pi}{L}$

$$\therefore \quad \boxed{h(y) = C_1 e^{\frac{n\pi}{L}y} + C_2 e^{-\frac{n\pi}{L}y}.}$$

$\left\{ e^{\frac{n\pi}{L}y}, e^{-\frac{n\pi}{L}y} \right\}$ lin. indep. solutions

also $\left\{ \cosh\left(\frac{n\pi}{L}y\right), \sinh\left(\frac{n\pi}{L}y\right) \right\}$ lin. indep. Solutions

also $\left\{ \cosh\left(\frac{n\pi}{L}(y-H)\right), \sinh\left(\frac{n\pi}{L}(y-H)\right) \right\}$ lin. indep. tions.

This last set is more convenient for the B.C. of the present problem. Thus,

$$\boxed{h(y) = C_1 \cosh\left(\frac{n\pi}{L}(y-H)\right) + C_2 \sinh\left(\frac{n\pi}{L}(y-H)\right)}$$

B.C. $h(H) = 0 \Rightarrow C_1 \cosh\left(\frac{n\pi}{L}(H-H)\right) + C_2 \underbrace{\sinh(0)}_{=0} = \boxed{C_1 = 0}$

$$\Rightarrow \quad \boxed{h(y) = C_2 \sinh\left(\frac{n\pi}{L}(y-H)\right).}$$

Product solutions:

$$\boxed{\left(u_1\right)_n(x,y) = B_n \underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{\text{oscillates}} \underbrace{\sinh\left(\frac{n\pi}{L}(y-H)\right)}_{\text{exponential behavior}}}$$

Superposition Pple:

$$\boxed{u_1(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}(y-H)\right)}$$

Using I.C.:

$$f_1(x) = u_1(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}(-H)\right)$$

Using orthogonality:

$$\int_0^L \left[\sum_{n=1} B_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}(-H)\right)\right] dx =$$

$$= \int_0^L f_1(x) \sin \frac{m\pi}{L}x \, dx$$

Interchanging $\sum$ and $\int$

$$\sum_{n=1}^{\infty} B_n \sinh\left(-\frac{n\pi H}{L}\right) \overbrace{\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx}^{=\begin{cases}0, & \text{if } n\neq m\\ \frac{L}{2}, & \text{if } n=m\end{cases}} =$$

$$= \int_0^L f_1(x)\sin\frac{m\pi}{L}x\,dx$$

$\Rightarrow$

$$B_n \frac{L}{2} \sinh\left(\frac{-n\pi H}{L}\right) = \int_0^L f_1(x)\sin\frac{n\pi}{L}x\,dx.$$

$\Rightarrow$

$$\boxed{B_n = \frac{2}{L\sinh\left(-\frac{n\pi H}{L}\right)}\int_0^1 f(x)\sin\left(\frac{n\pi}{L}x\right)dx.}$$

Laplace's Equation for a circular disk.

$$\begin{cases}\nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0. & 0<r<a,\ -\pi<\theta<\pi\\ u(a,\theta) = f(\theta), & -\pi<\theta<\pi\end{cases}$$

+ more conditions ?