CHAPTER 5.

Sturm - Liouville Eigenvalue Problem.

Consider the 1^{st} midterm heat conduction problem

$$u_t = K u_{xx}, \quad 0<x<L, \quad t>0.$$

$$u(0,t)=0, \quad u_x(L,t)=0, \quad u(x,0)=f(x)=\begin{cases} 1, & 0<x\le L/2 \\ 2, & L/2<x<L \end{cases}$$

Eigenvalue problem: $\begin{cases} \phi''(x)+\lambda\phi(x)=0 \\ \phi(0)=0, \quad \phi'(L)=0 \end{cases}$

Eigenvalues:

$$\lambda_n = \frac{(2n-1)\pi^2}{4L^2}$$

Set of eigenfunctions: $\{\phi_n\}_{n=1}^{\infty} = \left\{ \sin\left[\frac{(2n-1)\pi}{2L}x\right]\right\}$

In order to use separation of variables or eigenfunctions expansions, we assumed that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi}{2L}x\right) =$$

$$= \sum_{n=1}^{\infty} A_n \sin\left[\frac{(n-\frac{1}{2})\pi}{L}x\right]$$

How do we know that a piecewise smooth function as $f(x)$ can be expanded as this sin's series which doesn't have all the terms $\sin\frac{n\pi}{L}x$, $n=1,2\ldots$?

Consider examples.

(I) Circularly Symmetric Heat Flow. (2-D)

$$u_t = \kappa\,\nabla^2_{r,\theta}u = \kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}}\right]$$

or $$\boxed{u_t = \kappa\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}, \quad a<r<b,\ t>0$$

with B.C. $u(a,t)=0$, $u(b,t)=0$

I.C. $u(r,0)=f(r)$

Use separation of variables to find eigenfunctions

$$u(r,t)=\phi(r)\,h(t)$$

Two ODE are obtained

$$\boxed{\frac{dh}{dt}=-\lambda\kappa h} \quad \text{and} \quad \boxed{\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+\lambda r\phi=0}$$

Eigenvalue problem:

$$\begin{cases} \dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right) + \lambda r\phi = 0. \\ \phi(a) = 0, \quad \phi(b) = 0 \end{cases}$$

Eigenvalues : $\lambda_n = ?$, Eigenfunctions $\phi_n = ?$
$n = 1, 2, \ldots$

If IBVP is defined inside the complete circle.

$$\begin{cases} u_t = k\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right), & 0 < r < a \\ & t > 0 \\ u(a,\theta) = 0 \\ u(r,0) = f(r) \end{cases}$$

what's the eigenvalue problem?

$$\begin{cases} \dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right) + \lambda r\phi = 0 \\ \phi(a) = 0 \quad \text{and} \quad |\phi(0)| < \infty. \end{cases}$$

Eigenvalues = ? , Eigenfns = ?

II.- Heat flow in a nonuniform rod.

Problem # 5.4.1

$$\begin{cases} c\rho \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(K_0 \dfrac{\partial u}{\partial x}\right) + \alpha u, & 0<x<L, \ t>0 \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Separation of variables leads to

$$\frac{dh}{dt} = -\lambda h$$

and the eigenvalue problem:

$$\begin{cases} \dfrac{d}{dx}\left(K_0 \dfrac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\,\phi = 0 \\ \phi(0) = 0, \quad \phi(L) = 0 \end{cases}$$

Eigenfns. ?, Eigenvalues ?.

5.3 Sturm-Liouville Eigenvalue Problems.

All previous equations defining our eigenvalue problems are particular cases of the general ODE.

$$\boxed{\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right]+q(x)\phi(x)+\lambda\sigma\phi(x)=0} \qquad (5.1)$$

$$a<x<b.$$

Where λ is an eigenvalue.

Equation (5.1) is known as a Sturm-Liouville differential equation.

Follow with pages #162 : Boundary Conditions of various types.

and page #163: Statement of theorems.

3. *Vibrations of a nonuniform string*: $T_0 \dfrac{d^2\phi}{dx^2} + \alpha\phi + \lambda\rho_0\phi = 0$; in which case, $p = T_0$ (constant), $q = \alpha, \sigma = \rho_0$ (see Exercise 5.3.1).
4. *Circularly symmetric heat flow*: $\dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right) + \lambda r\phi = 0$, here the independent variable $x = r$ and $p(x) = x, q(x) = 0, \sigma(x) = x$.

Many interesting results are known concerning any equation in the form (5.3.1). Equation (5.3.1) is known as a **Sturm-Liouville differential equation**, named after two famous mathematicians active in the mid-1800s who studied it.

Boundary conditions. The linear homogeneous boundary conditions that we have studied are of the form to follow. We also introduce some mathematical terminology:

	Heat flow	Vibrating string	Mathematical terminology
$\phi = 0$	Fixed (zero) temperature	Fixed (zero) displacement	First kind or Dirichlet condition
$\dfrac{d\phi}{dx} = 0$	Insulated	Free	Second kind or Neumann condition
$\dfrac{d\phi}{dx} = \pm h\phi$ $\left(\begin{array}{l}+\text{left end}\\-\text{right end}\end{array}\right)$	(Homogeneous) Newton's law of cooling 0° outside temperature, $h = H/K_0$, $h > 0$ (physical)	(Homogeneous) elastic boundary condition $h = k/T_0$, $h > 0$ (physical)	Third kind or Robin condition
$\phi(-L) = \phi(L)$ $\dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L)$	Perfect thermal contact	—	Periodicity condition (example of mixed type)
$\lvert\phi(0)\rvert < \infty$	Bounded temperature	—	Singularity condition

5.3.2 Regular Sturm-Liouville Eigenvalue Problem

A regular Sturm-Liouville eigenvalue problem consists of the Sturm-Liouville differential equation,

$$\boxed{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0} \qquad a < x < b, \tag{5.3.2}$$

subject to the boundary conditions that we have discussed (excluding periodic and singular cases):

$$\boxed{\begin{aligned} \beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) &= 0 \\ \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) &= 0, \end{aligned}} \tag{5.3.3}$$

where β_i are real. In addition, to be called regular, the coefficients $p, q,$ and σ must be real and continuous everywhere (including the end points) and $p > 0$ and $\sigma > 0$ everywhere (also including the endpoints). For the regular Sturm-Liouville eigenvalue problem, many important general theorems exist. In Sec. 5.5 we will prove these results, and in Secs. 5.7 and 5.8 we will develop some more interesting examples that illustrate the significance of the general theorems.

Statement of theorems. At first let us just state (in one place) all the theorems we will discuss more fully later (and in some cases prove). For any regular Sturm-Liouville problem, all of the following theorems are valid:

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2 \sigma\,dx},$$
where the boundary conditions may somewhat simplify this expression.

<u>Definition</u> (Regular Sturm-Liouville Eigenvalue Problem).
(SLEP)

Sturm-Liouville differential equation:

$$\begin{cases} \dfrac{d}{dx}\left(p(x)\dfrac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a<x<b. \quad (1) \\ \text{B.C's.} \\ \beta_1\phi(a) + \beta_2\phi'(a) = 0 \quad (2) \\ \beta_3\phi(b) + \beta_4\phi'(b) = 0. \quad (3) \end{cases}$$

$\beta_2=\beta_4=0 \to$ DIRICHLET
$\beta_1\neq 0, \beta_3\neq 0$

$\beta_2\neq 0, \beta_4\neq 0 \Rightarrow$ Neumann
$\beta_1=0, \beta_3=0$

$\beta_i\neq 0$, for all i.

The eigenvalue problem (1)-(3) is called regular if $p(x)$, $q(x)$, and $\sigma(x)$ are real and continuous on $[a,b]$. Also, if $p(x)>0$ and $\sigma(x)>0$ on $[a,b]$.

Eigenvalue problems with periodicity condition or singularity cond. are not regular SLEP.

periodicity condition:
$$\phi(-L)=\phi(L)$$
$$\phi'(-L)=\phi'(L)$$

singularity cond.:
$$|\phi(0)|<\infty.$$

Importance of theorems for SLEP.

The most important of all these theorems is the one that establishes the completeness of the eigenfunctions. It means any p.w.s function can be represented by

$$f(x) \sim \underbrace{\sum_{n=1}^{\infty} a_n \phi_n(x)}_{\substack{\text{"Generalized Fourier Series"} \\ \text{of Eigenfunctions.}}}$$

Which converges to $[f(x^+) + f(x^-)]/2$ for any $a<x<b$.

The coefficients "a_n" can easily be determined using the orthogonality condition (Stat. #5).

$$\int_a^b \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx = 0, \qquad \lambda_n \neq \lambda_m.$$

Note this new "factor" or weight function.

In fact, multiplying $f(x)$ by $\sigma(x)\, \phi_m(x)$ and $\int_a^b dx$

$$\int_a^b f(x)\, \phi_m(x)\, \sigma(x)\, dx = \int_a^b \left[\sum_{n=1}^{\infty} a_n\, \phi_n(x)\right] \phi_m(x)\, \sigma(x)\, dx$$

$$\int_a^b f(x)\phi_m(x)\sigma(x)\,dx = \sum_{n=1}^{\infty} a_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx$$

$$= a_m \int_a^b \phi_m^2(x)\,\sigma(x)\,dx$$

$$\Rightarrow \quad \boxed{a_m = \frac{\int_a^b f(x)\phi_m(x)\sigma(x)\,dx}{\int_a^b \phi_m^2(x)\,\sigma(x)\,dx}} \qquad m = 1, 2, \dots$$

Derivation of Rayleigh quotient:

$$\boxed{\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b \left[p\,(d\phi/dx)^2 - q\,\phi^2\right] dx}{\int_a^b \phi^2(x)\,\sigma(x)\,dx.}} \qquad (8.1)$$

Multiplying equ. (1) by $\phi(x)$ and integrating $\int_a^b dx$

$$\int_a^b \left[\phi(x)\,\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\,\phi^2(x)\right] dx + \lambda \int_a^b \phi^2(x)\,\sigma(x)\,dx = 0.$$

$$\Rightarrow \quad \lambda = \frac{-\int_a^b \left[\phi(x)\,\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\,\phi^2(x)\right] dx}{\int_a^b \phi^2(x)\,\sigma(x)\,dx.}$$

Integrating by parts the first term in the numerator

$$\boxed{\lambda = \frac{-p(x)\phi(x)\frac{d\phi}{dx}\Big|_a^b + \int_a^b\left[p(x)\left(\frac{d\phi}{dx}\right)^2 - q(x)\phi^2(x)\right]dx}{\int_a^b \phi^2(x)\sigma(x)\,dx}}$$

Same as (8.1).

<u>Application of Rayleigh quotient:</u>

Consider the eigenvalue problem:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0.$$

To obtain the eigenvalues, we needed to consider all cases i) $\lambda > 0$, ii) $\lambda < 0$, and iii) $\lambda = 0$.

and finally we obtained that the only possibility was $\lambda > 0$.

The fact that $\lambda > 0$ is the only possibility can be obtained directly from Rayleigh Quotient. For this case $p(x) \equiv 1$
$q(x) \equiv 0$, $\sigma(x) \equiv 1$.

then

$$\lambda = \frac{\overset{0}{\cancel{\phi(x)\phi'(x)\Big|_a^b}} + \int_0^L [\phi'(x)]^2 dx}{\int_0^L \phi^2(x)\,dx} = \frac{\int_0^L [\phi'(x)]^2 dx}{\int_0^L \phi^2(x)\,dx} \qquad (9.2)$$

From (9.2), $\underline{\underline{\lambda \geq 0}}$

But also, if $\lambda = 0 \Leftrightarrow \phi'(x) \equiv 0$.

$\Rightarrow$ $\phi(x) \equiv$ Const. on $[0, L]$. Now, $\phi(0) = 0$, $\phi(L) = 0$

$\Rightarrow$ $\phi(x) \equiv 0$ and $\phi(x)$ can't be an eigenfunction!

Therefore, $\lambda = 0$ is not an eigenvalue and the only possibility is $\boxed{\lambda > 0}$.

5.4 Back to problem 5.4.1

$$\begin{cases} c(x)\rho(x)\, u_t = \dfrac{\partial}{\partial x}\left[k_0(x)\dfrac{\partial u}{\partial x}\right] + \alpha(x) u & 0 < x < L,\ t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$\alpha(x) < 0$.
"Restoring force"

Separation of Variables:

$u(x,t) = h(t)\,\phi(x)$.

$$c\rho\, h'\phi = h\,\frac{d}{dx}\left(k_0 \frac{d\phi}{dx}\right) + \alpha h \phi$$

Dividing by $c\rho h\phi$

$$\frac{h'}{h}(t) = \left[\frac{1}{c\rho\phi}\,\frac{d}{dx}\left(k_0\frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho}\right](x) = -\lambda$$

$\lambda > 0$ for exponentially decaying solutions

Therefore, the separation leads to

$$h'(t) = \lambda\, h(t) \qquad (11.1)$$

and the eigenvalue problem:

$$\begin{cases} \dfrac{d}{dx}\left[K_0(x) \dfrac{d\phi}{dx} \right] + \alpha(x)\phi + \lambda\, c(x)\rho(x)\, \phi = 0 & (11.2) \\ \phi(0) = 0, \quad \phi(L) = 0 & (11.3) \end{cases}$$

From B.C's.

Equations (11.2)-(11.3) form a regular Sturm-Liouville eigenvalue problem.

therefore, there are an infinite set of eigenvalues λ_n and the corresponding eigenfunctions form a complete set $\{\phi_n\}_{n=1}^{\infty}$.

The solutions for (11.1) are given by

$$h(t) = e^{-\lambda_n t}$$

By applying the pple of Superposition, we can express the solution of problem (5.4.1) as

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}$$

The coeffcients "a_n" can be determined using the orthogonality satisfied by the eigenfunctions ϕ_n. and the I.C.

In fact,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Multiplying by $c(x)\rho(x)\phi_m(x)$ and $\int_0^L dx$

$$\int_0^L f(x)\phi_m(x)\,c(x)\rho(x)\,dx = \int_0^L \left(\sum_{n=1}^{\infty} a_n \phi_n(x)\right)$$

$$= \sum_{n=1}^{\infty} a_n \int_0^L \phi_n(x)\phi_m(x)\,c(x)\rho(x)\,dx \overset{\text{Orthog.}}{=} a_m \int \phi_m^2(x)c(x)\rho(x)dx$$

$$\Rightarrow \boxed{a_m = \frac{\int_0^L f(x)\phi_m(x)c(x)\rho(x)\,dx}{\int_0^L \phi_m^2(x)\,c(x)\rho(x)dx}}, \qquad m = 1, 2, \ldots$$

If $f(x) > 0$ ~~on~~ part ~~of~~ $[0,L]$ and is non-negative on $[0,L]$, then

$$a_1 = \frac{\int_0^L f(x)\phi_1(x)c(x)\rho(x)dx}{\int_0^L \phi_1^2(x)c(x)\rho(x)\,dx} \neq 0$$

Since, $\phi_1(x) \neq 0$ on $(0,L)$ ($\phi_1(x)$ has no zeros on $(0,L)$). Thus, it is positive or negative in the entire interval. Also, $c(x), \rho(x) > 0$ on $[0,L]$ (regular SLEP).

$$\Rightarrow \int_0^L f(x)\phi_1(x)\,c(x)\rho(x)dx \neq 0 \Rightarrow a_1 \neq 0$$

For large t $(t \to \infty)$,

$$u(x,t) \simeq a_1 \phi_1(x) e^{-\lambda_1 t}$$

Thus, to obtain an approximation of the soln. for large t, the first eigenvalue λ_1 and the first eigenfunction $\phi_1(x)$ need to be approximated or computed numerically.

<u>Prove that eigenvalues λ_n are positive</u>

Substituting in Rayleigh quotient

$$\lambda = \frac{-p\phi\phi'\big|_0^L + \int_0^L [p(d\phi/dx)^2 - q\phi^2]dx}{\int_0^L \phi_m^2(x)\sigma(x)dx}$$

$p(x) = K_0(x)$, $q(x) = \alpha(x)$, $\sigma(x) = c(x)\rho(x)$, and using B.C's.

$$\Rightarrow \quad \lambda = \frac{\int_0^L [K_0(x)[\phi'(x)]^2 - \alpha(x)\phi^2(x)]dx}{\int_0^L \phi^2(x)\, c(x)\rho(x)dx}$$

Since $K_0(x) > 0$ and $\alpha(x) < 0$ on $(0,L)$ $\Rightarrow \lambda \geq 0$

But $\lambda = 0 \Leftrightarrow \phi'(x) \equiv 0$, and $\phi(x) \equiv 0 \Rightarrow$ $\phi(x)$ is not eigenfunction Contradiction!

$$\Rightarrow \quad \boxed{\lambda > 0}$$

Therefore, $\lim\limits_{t\to\infty} u(x,t) = \lim\limits_{t\to\infty} \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t} \longrightarrow 0$