## 5.5 Self-Adjoint Operators and Sturm – Liouville EVP.

Consider

$$I = \int_a^b v(x)\left[u''(x) + 4u(x)\right] dx$$

Using integration by parts twice (assuming $v(x)$ and $u(x)$ suff. smooth)

$$\int_a^b v(x)\left[u''(x)+4u(x)\right] dx = \int_a^b v(x)\,u''(x)\,dx + 4\int_a^b v(x)u(x)\,dx =$$

$$= v(x)\,u'(x)\Big|_a^b - \int_a^b v'(x)\,u'(x)\,dx + 4\int_a^b v(x)\,u(x)\,dx =$$

$$= vu'\Big|_a^b - uv'\Big|_a^b + \int_a^b u\,v''\,dx + 4\int_a^b vu\,dx$$

$$= \left(vu' - uv'\right)\Big|_a^b + \int_a^b u\left[v'' + 4v\right] dx$$

$$\Rightarrow \boxed{\int_a^b \left[v\left[u''+4u\right] - u\left[v''+4v\right]\right] dx = \left(vu'-uv'\right)\Big|_a^b} \quad (1.1)$$

Defining $L[u] \equiv \dfrac{d^2u}{dx^2} + 4u$, where $L \equiv \dfrac{d^2}{dx^2} + 4I$.

DIFFERENTIAL OPERATOR.

The last expression can be written as

$$\boxed{\int_a^b \left(v\,L[u] - u\,L[v]\right) dx = \left[vu' - uv'\right]\Big|_a^b} \quad (1.2)$$

Consider now

$$I = \int_a^b v(x)\left[u''(x) + u'(x) + u\right] dx$$

Using integration by parts (assuming $v(x)$ and $u(x)$ suff. smooth)

$$\int_a^b v[u''+u'+u]\,dx = \int_a^b vu''\,dx + \int_a^b vu'\,dx + \int_a^b vu\,dx =$$

$$= vu'\Big|_a^b - \int v'u'\,dx + vu\Big|_a^b - \int uv'\,dx + \int_a^b vu\,dx =$$

$$= vu'\Big|_a^b - uv'\Big|_a^b + \int uv''\,dx + vu\Big|_a^b - \int uv'\,dx + \int_a^b vu\,dx$$

$$= \int_a^b u\,[v'' - v' + v]\,dx + (vu' - uv')\Big|_a^b + vu\Big|_a^b$$

$$\therefore \quad \boxed{\int_a^b \Big[v\,[u''+u'+u] - u\,[v''-v'+v]\Big]dx = \Big[vu' - uv'\Big]\Big|_a^b + vu\Big|_a^b} \quad (2.1)$$

or defining the differential operator:

$$\hat{L}\,[v] \equiv v'' - v' + v \quad \text{and} \quad L\,[u] \equiv u'' + u' + u.$$

we obtain

$$\boxed{\int_a^b \Big[v\,L[u] - u\,\hat{L}\,[v]\Big]dx = \Big[u'v - uv'\Big]\Big|_a^b + vu\Big|_a^b} \quad (2.2)$$

There is an important difference between the two integral formulas (1.2) and (2.2).
That is in the differential operator obtained after integrating by part twice. (in a way that all derivatives are removed from the function $u(x)$)

i) In (1.2) the new differential operator $\hat{L}$ acting over the function $v(x)$, is such that

$$\hat{L} = L.$$

ii) But in (2.1) this new differential operator $\hat{L}$ acting over $v(x)$ is

$$\hat{L} \equiv \frac{d^2}{dx^2} - \frac{d}{dx} + I \neq \frac{d^2}{dx^2} + \frac{d}{dx} + I = L$$

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The operator : $L[u] = u'' + 4u$ is a particular case of the Sturm-Liouville operator

$$L[u] \equiv \left[p(x)u'(x)\right]' + q(x)\,u(x).$$

Where $p(x) \equiv 1$, $q(x) \equiv 4$.

However, the operator

$$\hat{L}[u] = u'' - u' + u$$

is not of Sturm-Liouville type.

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GREEN'S FORMULA.

Theorem 1:- If $L[u] \equiv [p(x)u'(x)]' + q(x)u(x)$ is the Sturm-Liouville operator and $u(x), v \in C^2[a,b]$ and $p(x)$ is continuous then

$$\int_a^b [uL[v] - vL[u]]dx = p(x)\left[uv' - vu'\right]\Big|_a^b$$

Proof.-

$$\int_a^b \left[u[(pv')' + \cancel{qv}] - v[(pu')' + \cancel{qu}]\right]dx =$$

$$= \int_a^b [u(pv')' - v(pu')']dx =$$

$$= puv'\Big|_a^b - \int_a^b \cancel{pv'u'}dx - pu'v\Big|_a^b + \int_a^b \cancel{pu'v'}dx =$$

$$= p(x)(uv' - u'v)\Big|_a^b$$

$$\therefore \quad \boxed{\int_a^b [uL[v] - vL[u]]dx = p(x)[uv' - u'v]\Big|_a^b} \quad \checkmark \quad (4.1)$$

This is called Green's formula

The differential form of this formula is called Lagrange identity:

$$\boxed{uL[v] - vL[u] = \left[p(x)\left(uv' - u'v\right)\right]'} \quad (5.1)$$

Definition.- If an operator $L$ Satisfies (4.1) for any two functions $u(x), v(x) \in C^2[a,b]$ and $u$ and $v$ satisfy boundary conditions Such that the right hand side of (4.1) is zero

or $$\boxed{\int_a^b [uL[v] - vL[u]]\,dx = 0} \quad (5.2)$$

then we will say that the operator $L$ (Corresponding to these boundary conditions) is Self-Adjoint.

Remark 1.- If $L$ is such that $\hat{L}$ (obtained by int. twice $\int_a^b vL[u]dx$) is different than $L$ and the BC's cancels the rhs, then $\hat{L}$ is called the adjoint of $L$ for these BC's.

$(L = \hat{L})$

Remark 2.- A Self-adjoint operator $L$ is a Concept Similar to the concept of Symmetric matrix: $A = A^T$

**Theorem 2** - Consider the regular S-L EVP

$$\begin{cases} L[\phi] + \lambda\sigma(x)\phi = [p(x)\phi']' + q(x)\phi + \lambda\sigma(x)\phi = 0 & (6.1) \\ \beta_1\phi(a) + \beta_2\phi'(a) = 0 & (6.2) \\ \beta_3\phi(b) + \beta_4\phi'(b) = 0 & (6.3) \end{cases}$$

The operator $L$ defining (6.1) is self-adjoint for the boundary conditions (6.2) and (6.3): Dirichlet, Neumann, Robin or combinations of them.

**Proof.** - In the Homework problems.

## Application of the concept of Self-adjoint operator

## Orthogonality of eigenfunctions for the SL-EVP.

Let $Lu \equiv [p(x)u']' + q(x)u$. be the S-L operator and $\phi_n$ and $\phi_m$ two eigenfunctions corresponding to eigenvalues $\lambda_n$ and $\lambda_m$ with $\lambda_n \neq \lambda_m$.

Then,

$$\int_a^b \left([\phi_m L[\phi_n] - \phi_n L[\phi_m]\right)dx \overset{\text{Thm 1}}{=} p(x)\left(\phi_m\phi_n' - \phi_n\phi_m'\right)\Big|_a^b \underset{\text{Thm 2}}{\overset{\text{Previous}}{=}} 0$$

Therefore,

$$0 = \int_a^b \left[\phi_m\left(-\lambda_n \sigma \phi_n\right) - \phi_n\left(-\lambda_m \sigma \phi_m\right)\right] dx =$$

$$= \left(\lambda_m - \lambda_n\right) \int_a^b \phi_n \phi_m \sigma(x)\, dx$$

Since $\lambda_m \neq \lambda_n$ then,

$$\int_a^b \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx = 0.$$

or $\phi_n(x)$ is orthogonal to $\phi_m(x)$ with weight $\sigma(x)$.

<u>Thm.</u>- All the eigenvalues for a regular SL-EVP are real.

<u>Proof.</u>-

First, we will prove that if $\lambda$ is a complex eigenvalue with corresponding eigenfunction $\phi$, then $\bar{\lambda}$ is also an eigenvalue with corresponding eigenfunction $\bar{\phi}$. In fact, if

$$L[\phi] + \lambda \sigma \phi = 0$$

$\Rightarrow$ the conjugate equation is $\qquad z = x+iy \Rightarrow \bar{z} = x - iy.$

$$\overline{L[\phi]} + \bar{\lambda}\sigma\bar{\phi} = 0, \qquad (\sigma \text{ is real})$$

Now, $\overline{L[\phi]} = L[\bar{\phi}]$ (why?) Ex. 5.5.7. $\Rightarrow \boxed{L(\bar{\phi}) + \bar{\lambda}\sigma\bar{\phi} = 0}$ (8.1)

and $\bar{\phi}$ satisfies the same BC's that $\phi$ satisfies

$$\beta_1 \bar{\phi}(a) + \beta_2 \bar{\phi}'(a) = 0, \qquad \beta_1, \beta_2 \text{ are real.}$$

and $\beta_3 \bar{\phi}(b) + \beta_4 \bar{\phi}'(b) = 0$

Therefore, $\bar{\phi}$ satisfies a SL-EVP. with eigenvalue $\bar{\lambda}$.

On the other hand, $\lambda \neq \bar{\lambda}$ and the orthogonality condition is verified

$$(\lambda - \bar{\lambda}) \int_a^b \phi \bar{\phi} \sigma \, dx = (\lambda - \bar{\lambda}) \int_a^b |\phi|^2 \sigma \, dx = 0$$

$\Rightarrow \phi(x) \equiv 0$ Contradiction. Then $\lambda$ can't be complex.

<u>Thm.</u>- For each eigenvalue $\lambda_n$ there is only one eigenfunction linearly independent.

<u>Proof.</u>- Assume $\phi_1(x)$ and $\phi_2(x)$ are eigenfunctions corresponding to the same eigenvalue $\lambda$ then,

$$\begin{array}{l} \phi_2 * \left[ L(\phi_1) + \lambda \sigma \phi_1 = 0 \right] \\ \phi_1 * \left[ L(\phi_2) + \lambda \sigma \phi_2 = 0 \right] \\ \hline \phi_2 L(\phi_1) - \phi_1 L(\phi_2) = 0 \end{array}$$

Using Lagrange identity

$$0 = \phi_2 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx}\left[ p\left(\phi_2 \phi_1' - \phi_1 \phi_2'\right)\right]$$

$$\Rightarrow p\left(\phi_2 \phi_1' - \phi_1 \phi_2'\right) \quad \text{constant}$$

Clearly, if
$$\begin{cases} \phi_1(a) = 0, \quad \phi_2(a) = 0 \\ \qquad\quad \text{or} \\ \phi_1'(a) = 0, \quad \phi_2'(a) = 0 \end{cases} \quad \text{or} \quad \begin{array}{c} \phi_1(b) = 0, \quad \phi_2(b) = 0 \\ \text{or} \\ \phi_1'(b) = 0, \quad \phi_2'(b) = 0 \end{array}$$

then
$$\left.\begin{array}{c} \left(\phi_2 \phi_1' - \phi_1 \phi_2'\right)(a) = 0 \\ \text{or} \\ \left(\phi_2 \phi_1' - \phi_1 \phi_2'\right)(b) = 0 \end{array}\right\} \Rightarrow \left(\phi_2 \phi_1' - \phi_1 \phi_2'\right)(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\phi_2}{\phi_1}\right) = 0$$

$$\Rightarrow \phi_2 = C\phi_1 \Rightarrow \phi_2 \text{ and } \phi_1 \text{ are linearly independent.}$$

For Robin B.C. the thm is also true.