

Vibration of Highly-stretched membrane:

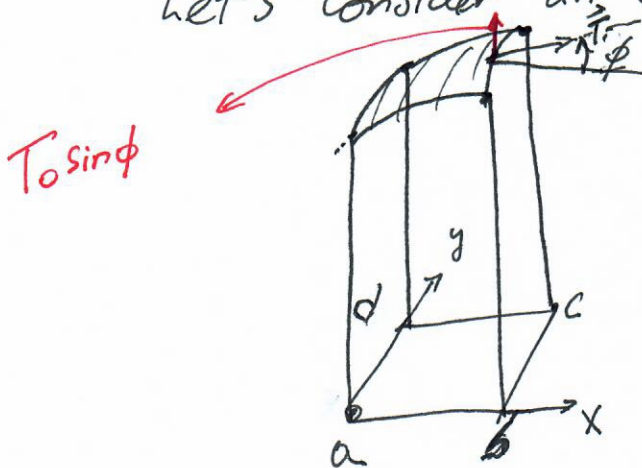
Two-dimensional Wave Equation.

Assumptions:

- i) Displacement is only vertical
given by $u(x, y, t)$. This is a
consequence that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are small.
magnitude
- ii) Also, tensile forces are constant: $T_0 = |\vec{T}|$.
- iii) the tensile forces are tangent to the
Surface of the membrane.
They form an angle ϕ with the xy -plane.

ρ_0 : mass density per unit of area.

Let's consider an arbitrary ^{rectangular} portion of the membrane



The ^{vertical} force at any
boundary point is

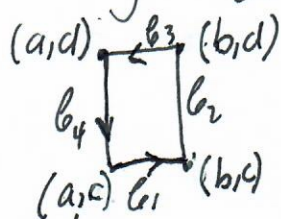
$$T_0 \sin \phi(x, y, t)$$

$$\approx T_0 \tan \phi = T_0 \frac{\partial u}{\partial x}(x, y, t).$$

Newton law applied to this portion of the membrane

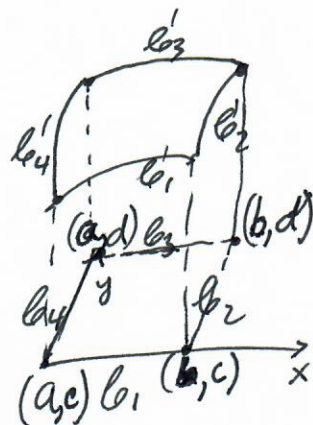
$$\int_a^b \int_c^d \rho_0 \frac{\partial^2 u}{\partial t^2}(x, y, t) dy dx$$

$$= \int T_0 \frac{\partial u}{\partial n} dl$$



$$= \int_{l_1}^b T_0 \left(-\frac{\partial u}{\partial y}\right)(x, c, t) dx + \int_{l_2}^d T_0 \frac{\partial u}{\partial x}(b, y, t) dy$$

$$+ \int_{l_3}^a T_0 \frac{\partial u}{\partial y}(x, d, t) (dx) + \int_{l_4}^c T_0 \left(\frac{\partial u}{\partial x}\right)(a, y, t) (dy)$$



$$= \int_a^b T_0 \left[\frac{\partial u}{\partial y}(x, d, t) - \frac{\partial u}{\partial y}(x, c, t) \right] dx$$

$$+ \int_c^d T_0 \left[\frac{\partial u}{\partial x}(b, y, t) - \frac{\partial u}{\partial x}(a, y, t) \right] dy$$

Fund.
Theor.
Calculus
=

$$\int_a^b T_0 \left[\int_c^d \frac{\partial}{\partial y} \left[\frac{\partial u}{\partial y} \right] (x, y, t) dy \right] dx$$

$$+ \int_c^d T_0 \left[\int_a^b \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial x} \right] (x, y, t) dx \right] dy$$

$$= \int_a^b \int_c^d T_0 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] (x, y, t) dy dx$$

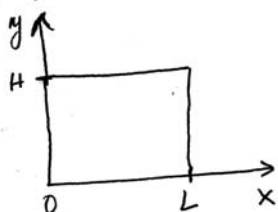
Finally,
$$\int_a^b \int_c^d \left[\rho_0 \frac{\partial^2 u}{\partial t^2} - T_0 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right] (x, y, t) dy dx = 0$$

Because the portion $[a, b] \times [c, d]$ of the membrane is arbitrary, then

$$\boxed{\begin{aligned} \rho_0 \frac{\partial^2 u}{\partial t^2} &= T_0 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ \text{or } u_{tt} &= C^2 (u_{xx} + u_{yy}) \end{aligned}} \quad C^2 \equiv \frac{T_0}{\rho_0}$$

7.3 Vibrating Rectangular Membrane.

IBVP:

$$\left\{ \begin{array}{l} \text{PDE: } u_{tt} = c^2 (u_{xx} + u_{yy}) = c^2 \nabla_{xy}^2 u \quad (1) \\ \text{BC's: } \begin{cases} u(0, y, t) = 0, & u(x, 0, t) = 0 \quad (2) \\ u(L, y, t) = 0, & u(x, H, t) = 0 \quad (3) \end{cases} \\ \text{IC's: } \begin{cases} u(x, y, 0) = \alpha(x, y) \quad (4) \\ u_t(x, y, 0) = \beta(x, y) \quad (5) \end{cases} \end{array} \right.$$


Separation of variables:

$$u(x, y, t) = h(t) \phi(x, y)$$

Substitution in (1) leads to

$$h'' \phi = c^2 h (\phi_{xx} + \phi_{yy}) \xrightarrow{\text{Dividing by } c^2 h \phi} \frac{h''}{c^2 h} = \frac{1}{\phi} (\phi_{xx} + \phi_{yy}) = -\lambda$$

Explain why?

Therefore, we obtain the two equations (ODE):

$$\begin{cases} h'' - c^2 \lambda h = 0 \\ \phi_{xx} + \phi_{yy} = -\lambda \phi \end{cases}$$

From the boundary conditions:

$$\begin{array}{l} \phi(0, y) h(t) = 0 \Rightarrow \phi(0, y) = 0 \\ \phi(L, y) h(t) = 0 \Rightarrow \phi(L, y) = 0 \\ \phi(x, 0) h(t) = 0 \Rightarrow \phi(x, 0) = 0 \\ \phi(x, H) h(t) = 0 \Rightarrow \phi(x, H) = 0 \end{array}$$

Two-dimensional eigenvalue problem:

$$\begin{cases} \nabla_{x,y}^2 \phi = \phi_{xx} + \phi_{yy} = -\lambda \phi & (2.1) \end{cases}$$

$$\begin{cases} \phi(0,y) = 0, \quad \phi(x,0) = 0 & (2.2) \end{cases}$$

$$\begin{cases} \phi(L,y) = 0, \quad \phi(x,H) = 0 & (2.3) \end{cases}$$

The equation for $h(t)$ has been solved for the 1-D problem

$$h''(t) + \lambda C h(t) = 0$$

Assuming $\lambda > 0$ (oscillatory solutions).

$$\boxed{h(t) = C_1 \cos c\sqrt{\lambda}t + C_2 \sin c\sqrt{\lambda}t} \quad (2.4)$$

The eigenvalue problem is another PDE with homogeneous boundary conditions (2.2)-(2.3).

We try a further separation of variables in the variables x, y

$$\phi(x,y) = f(x)g(y) \quad (2.5)$$

Substitution of (2.5) into (2.1) leads to

$$g(y) f''(x) + f(x) g''(y) = -\lambda f(x) g(y)$$

Dividing by $f(x)g(y)$.

$$\frac{f''(x)}{f} = -\lambda - \frac{g''(y)}{g(y)} = -\mu$$

Two ODE's result from this equation:

$$\boxed{f'' + \mu f = 0} \quad \text{and} \quad \boxed{g'' + (\lambda - \mu)g = 0}$$

And using the BC's ^{(2.2) and (2.3)}, we obtain two eigenvalue problems:

$$\boxed{\begin{aligned} f'' + \mu f &= 0 \\ f(0) &= 0, \quad f(L) = 0 \end{aligned}} \quad (3.1)$$

$$\text{and} \quad \boxed{\begin{aligned} g'' + (\lambda - \mu)g &= 0 \\ g(0) &= 0, \quad g(H) = 0 \end{aligned}} \quad (3.2)$$

Eigenvalues for (3.1) are $\boxed{\mu_n = \left(\frac{n\pi}{L}\right)^2}, n=1,2,\dots$ (3.3)

and eigenfunctions are $\boxed{f_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$ (3.4)

For each $n=1,2,\dots$ (3.2) defines an eigenvalue problem:

$$\begin{cases} g_n'' + (\lambda_n - \mu_n) g_n = 0 \\ g_n(0) = 0, \quad g_n(H) = 0 \end{cases}$$

whose eigenvalues are $\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2$, $m=1,2,\dots$ (4.1)

and corresponding eigenfunctions are

$$g_{nm}(y) = \sin\left(\frac{m\pi}{H}y\right) \quad (4.2)$$

Combining (3.3) and (4.1)

$$\lambda_{nm} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \quad \begin{matrix} n=1,2,\dots \\ m=1,2,\dots \end{matrix}$$

and we have obtained the eigenvalues of the EVP. (2.1)-(2.3).

From (2.5), (3.4) and (4.2), we also obtain the eigenfunctions:

$$\phi_{nm} = \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right) \quad \begin{matrix} n=1,2,\dots \\ m=1,2,\dots \end{matrix}$$

Principle of Superposition:

Two families of product solutions combine to form the soln:

$$\begin{aligned}
 U(x,t) = & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{H}y\right) \cos(c\sqrt{\lambda_{nm}}t) \\
 & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{nm} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right) \sin(c\sqrt{\lambda_{nm}}t)
 \end{aligned} \tag{5.1}$$

The coefficients A_{nm} and B_{nm} can be determined from the IC's

$$\overset{\text{known}}{\uparrow} \alpha(x,y) = U(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{H}y\right)$$

Double Fourier Series.

There is a theory for this double F.S. similar to single F.S. However, it can be treated as two iterated single F.S.

We will assume that $\alpha(x,y)$ and $\beta(x,y)$ are at least piecewise smooth in each variable.

In fact,

$$\alpha(x, y) = \sum_{m=1}^{\infty} \underbrace{\left(\sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) \right)}_{C_m(x)} \sin\left(\frac{m\pi}{H}y\right) \quad (6.1)$$

For x fixed,

$$C_m(x) = \sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) = \frac{2}{H} \int_0^H \underbrace{\alpha(x, y) \sin\left(\frac{m\pi}{H}y\right) dy}_{K(x)} \quad (6.2)$$

The equation (6.2) is the Fourier Sine Series of $K(x)$

then, $A_{nm} = \frac{2}{L} \int_0^L K(x) \sin\left(\frac{n\pi}{L}x\right) dx$

or

$$\begin{aligned} A_{nm} &= \frac{2}{L} \int_0^L \left[\frac{2}{H} \int_0^H \alpha(x, y) \sin\left(\frac{m\pi}{H}y\right) dy \right] \sin\frac{n\pi}{L}x \, dx \\ &= \frac{4}{LH} \int_0^L \int_0^H \alpha(x, y) \sin\left(\frac{m\pi}{H}y\right) \sin\left(\frac{n\pi}{L}x\right) dy \, dx. \end{aligned} \quad (6.3)$$

Double Integral

$$n=1, 2, \dots$$

$$m=1, 2, \dots$$

From $u_t(x, y, 0) = \beta(x, y)$

$$u_t(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{nm} C \sqrt{\lambda_{nm}} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right) = \beta(x, y)$$

$$\Rightarrow \boxed{C \sqrt{\lambda_{nm}} B_{nm} = \frac{4}{LH} \int_0^L \int_0^H \beta(x, y) \sin\left(\frac{m\pi}{H}y\right) \sin\left(\frac{n\pi}{L}x\right) dy dx} \quad (7.1)$$

$m=1, 2, \dots$
 $n=1, 2, \dots$

Summarizing, The solution of (4)-(5) is given by (5.1) with the coefficients A_{nm} and B_{nm} defined by (6.3) and (7.1).

All this separation process

$$u(x, y, t) = \phi(x, y) h(t)$$

and later $\phi(x, y) = f(x) g(y)$

is equivalent to start from the beginning with

$$u(x, y, t) = f(x) g(y) h(t).$$

Vibrating Rectangular membrane.

Separation of Variables:

$$u(x, y, t) = h(t) \phi(x, y).$$

$$h''(t) \phi(x, y) = c^2 \nabla^2 \phi(x, y)$$

$$\Rightarrow \frac{1}{c^2} \frac{h''(t)}{h(t)} = \frac{1}{\phi(x, y)} \nabla^2 \phi(x, y) = -\lambda$$

$\Rightarrow$ Two ODE's

$$h''(t) + \lambda c^2 h(t) = 0$$

2D - Eigenvalue problem:

$$\begin{cases} \nabla^2 \phi(x, y) + \lambda \phi(x, y) = 0 \\ \phi(x, y) = 0, \end{cases} \quad (x, y) \in \text{boundary rectang.}$$

Further separation: $\phi(x, y) = f(x) g(y).$

$$g(y) f''(x) + g''(y) f(x) + \lambda f(x) g(y) = 0$$

$$\Rightarrow \frac{f''(x)}{f(x)} = \frac{g''(y)}{g(y)} + \lambda g(y) = -\mu$$

1D - Eigenvalue problems:x - variable

$$\begin{cases} \frac{d^2 f}{dx^2} + \mu f(x) = 0 \\ f(0) = 0, f(1) = 0 \end{cases}$$

y - variable.

$$\begin{cases} \frac{d^2 g}{dy^2} + (\lambda - \mu) g(y) = 0 \\ g(0) = 0, g(1) = 0 \end{cases}$$

Eigenvalues and Eigenfunctions

$$\mu_n = (n\pi)^2, \quad n=1,2,\dots$$

$$f_n(x) = \sin(n\pi x), \quad n=1,2,\dots$$

Eigenvalues and Eigenfunctions.

$$(\lambda_n)_m = (n\pi)^2 + (m\pi)^2$$

$$g_m(y) = \sin(m\pi y)$$

2-D Eigenvalues and Eigenfunctions.

$$\lambda_{nm} = (n\pi)^2 + (m\pi)^2$$

$$n, m = 1, 2, \dots$$

$$\phi_{nm} = \sin(n\pi x) \sin(m\pi y)$$

MODES OF VIBRATIONS.

Consider the IBVP.

$$\begin{cases} u_{tt} = c^2 \nabla_{x,y}^2 u \\ u(0,y,t) = 0, \quad u(x,0,t) = 0 \\ u(L,y,t) = 0, \quad u(x,H,t) = 0 \\ u(x,y,0) = \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{H}y\right) \\ u_t(x,y,0) = 0 \end{cases}$$

Soln:

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{nm} \cos(c\sqrt{\lambda_{nm}} t) + B_{nm} \sin(c\sqrt{\lambda_{nm}} t) \right] * \\ * \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right)$$

Using I.C's:

$$u(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right) = \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{H}y\right)$$

Using orthogonality: $A_{11} = 1$, $A_{nm} = 0$, all other case.

From the 2nd I.C. $u_t(x,y,0) = 0 \Rightarrow B_{nm} = 0$, for all n,m .

$\therefore$

$$u(x,y,t) = \cos(c\sqrt{\lambda_{11}} t) \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{H}y\right)$$

The previous Soln. constitutes the most elementary mode of vibration. All other modes can be obtained

1) By defining initial conditions:

$$u(x, y, 0) = \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right). \quad \begin{matrix} n=1, 2, \dots \\ m=1, 2, \dots \end{matrix}$$

$$u_t(x, y, 0) = 0$$

or

$$2) \quad u(x, y, 0) = 0$$

$$u_t(x, y, 0) = \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right).$$
