7.4 Statements and Illustration of theorems for Eigenvalue Problem:

$$\begin{cases} \nabla^2\phi + \lambda\phi = 0, & \vec{x} = (x,y,z) \in \Omega \quad (1) \\ a\phi + b\,\nabla\phi\cdot\hat{n}(\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega \quad (2) \end{cases}$$

$a(\vec{x})$, $b(\vec{x})$. , $\hat{n}$: outer normal to $\partial\Omega$ at $\vec{x}_s$.

The equation (1) is called the Helmholtz equation.
This equation is a particular case of the Sturm-Liouville equation in two or three dimensions.

$$\nabla\cdot(p\nabla\phi) + q\phi + \lambda\sigma\phi = 0. \quad (3)$$

There are similar theorems for regular Sturm-Liouville in higher dimensions as we have proved previously in 1-D.
They are outlined in the book for the simpler EVP (1) - (2).

Show Book.

1. All the eigenvalues are real.
2. There exists an infinite number of eigenvalues. There is a smallest eigenvalue, but no largest one.

   lin. indep.
3. Corresponding to an eigenvalue, there *may* be many eigenfunctions (unlike regular Sturm-Liouville eigenvalue problems).
4. The eigenfunctions $\phi(x, y)$ form a "complete" set, meaning that any piecewise smooth function $f(x, y)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x,y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x,y). \tag{7.4.4}$$

   Here $\sum_\lambda a_\lambda \phi_\lambda$ means a linear combination of all the eigenfunctions. The series converges in the mean if the coefficients a_λ are chosen correctly.
5. Eigenfunctions belonging to different eigenvalues (λ_1 and λ_2) are orthogonal relative to the weight $\sigma(\sigma = 1)$ over the entire region R. Mathematically,

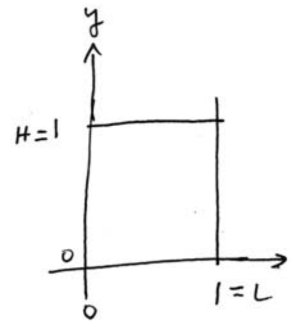
$$\iint\limits_R \phi_{\lambda_1}\phi_{\lambda_2}\, dx\, dy = 0 \quad \text{if } \lambda_1 \neq \lambda_2, \tag{7.4.5}$$

   where $\iint_R \; dx\, dy$ represents an integral over the entire region R. Furthermore, different eigenfunctions belonging to the same eigenvalue can be made orthogonal by the Gram-Schmidt process (see Sec. 7.5). Thus, we may assume that (7.4.5) is valid even if $\lambda_1 = \lambda_2$ as long as ϕ_{λ_1} is independent of ϕ_{λ_2}.
6. An eigenvalue λ can be related to the eigenfunction by the Rayleigh quotient:

$$\lambda = \frac{-\oint \phi \nabla\phi \cdot \hat{\boldsymbol{n}}\, ds + \iint_R |\nabla\phi|^2\, dx\, dy}{\iint_R \phi^2\, dx\, dy}. \tag{7.4.6}$$

   The boundary conditions often simplify the boundary integral.

For example, if $L=1$, $H=1$.



$$\lambda_{nm} = (n\pi)^2 + (m\pi)^2$$

$$\lambda_{12} = \pi^2 + 4\pi^2 = 5\pi^2.$$

$$\lambda_{21} = 4\pi^2 + \pi^2 = 5\pi^2 \quad \Rightarrow \quad \lambda_{12} = \lambda_{21}$$

$$\phi_{12} = \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{2\pi}{H}y\right)$$

$$\phi_{21} = \sin\left(\frac{2\pi}{L}x\right) \sin\left(\frac{\pi}{H}y\right)$$

It's easy to prove that $\phi_{12} \perp \phi_{21}$ (orthogonal).

In fact,

$$\int_0^L \int_0^H \phi_{12}(x,y)\, \phi_{21}(x,y)\, dy\, dx =$$

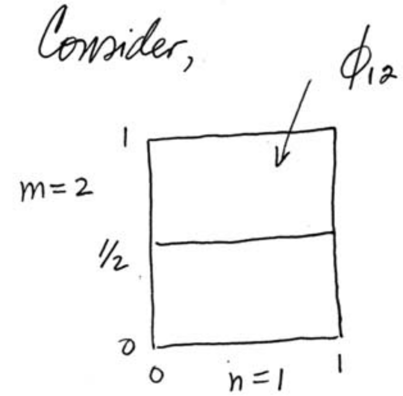
$$= \int_0^L \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{2\pi}{L}x\right) \overbrace{\left[\int_0^H \sin\left(\frac{2\pi}{H}y\right) \sin\left(\frac{\pi}{H}y\right) dy\right]}^{0} dx$$

$$= 0.$$

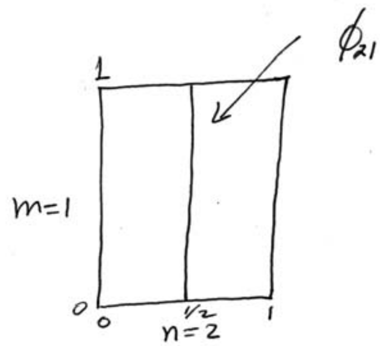
Thus, we have proved that for the same eigenvalue $\lambda_{12} = \lambda_{21} = 5\pi^2$, there are two linearly independent eigenfunctions: $\phi_{12}(x)$ and $\phi_{21}(x)$.

These fact of having two linearly indep. eigenfns. for the same eigenvalue, can be seen from a consideration of the size of the cells enclosed by the nodal curves.

Consider,



and



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Speed is the same in both which is equivalent to say that

$$\text{frequency (natural)} = \sqrt{\lambda_{12}}\, c = \sqrt{\lambda_{21}}\, c$$

$$\Rightarrow \lambda_{12} = \lambda_{21}.$$

Therefore, two different modes

$$\cos(\sqrt{\lambda_{12}}\, ct)\, \phi_{12}(x,y)$$

and

$$\cos(\sqrt{\lambda_{21}}\, ct)\, \phi_{21}(x,y)$$

have the same vibration freq.

5) Orthogonality can be proved easily from 1-D orthog.

$$\int_0^L \int_0^H \overbrace{\left[\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)\right]}^{\phi_{nm}(x,y)} \overbrace{\left[\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{Q\pi}{H}y\right)\right]}^{\phi_{kQ}} dy\,dx = 0$$

if $n \neq k$, and $m \neq Q$.

Again, It can be used to determine the coefficients of a generalized Fourier Series.

$$f(x,y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x,y).$$

multiply by ϕ_{λ_i}

$$a_{\lambda_i} = \frac{\iint_R f(x,y)\phi_{\lambda_i}(x,y)\,dx\,dy}{\iint_R \phi_{\lambda_i}^2(x,y)\,dx\,dy}$$

Convergence of Fourier Series (Generalized).

If $\{\phi_n\}_{n=1}^{\infty}$ are the eigenfunctions corresponding to the multidimensional (2 or 3-D) eigenvalue problem

$$\begin{cases} \nabla^2\phi + \lambda\phi = 0, & \vec{x} \in \Omega \\ [\beta_1 \phi + \beta_2 \nabla\phi \cdot \hat{n}](\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega. \end{cases}$$

Then, $\{\phi_n\}_{n=1}^{\infty}$ form a complete set for any $\begin{Bmatrix} 2\text{-D} \\ 3\text{-D} \end{Bmatrix}$ piecewise smooth function.

or $$f(\vec{x}) \sim \sum_{\lambda} a_\lambda \phi_\lambda(\vec{x})$$

where $$a_{\lambda_i} = \frac{\iiint_\Omega f(\vec{x})\, \phi_{\lambda_i}(\vec{x})\, dv}{\iiint_\Omega \phi_{\lambda_i}^2(\vec{x})\, dv}$$

The series converges in the L_2 norm. It means

$$E^2 = \iiint_\Omega \left[f(\vec{x}) - \sum_{\lambda}^{M} a_\lambda \phi_\lambda \right]^2 dv \xrightarrow[M\to\infty]{} 0$$

Rmk: In Sect 5.10 (1-D), it is proved that for a truncated series (finite linear comb. of the first M eigenfns, the least square error is minimum if a_λ are Fourier coeffs