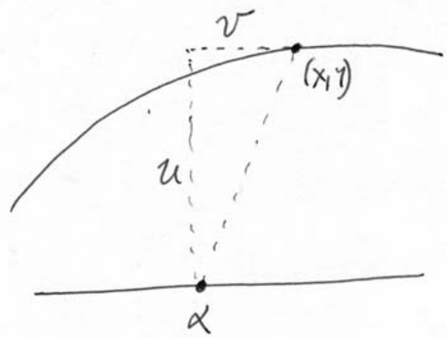


Vibrating String

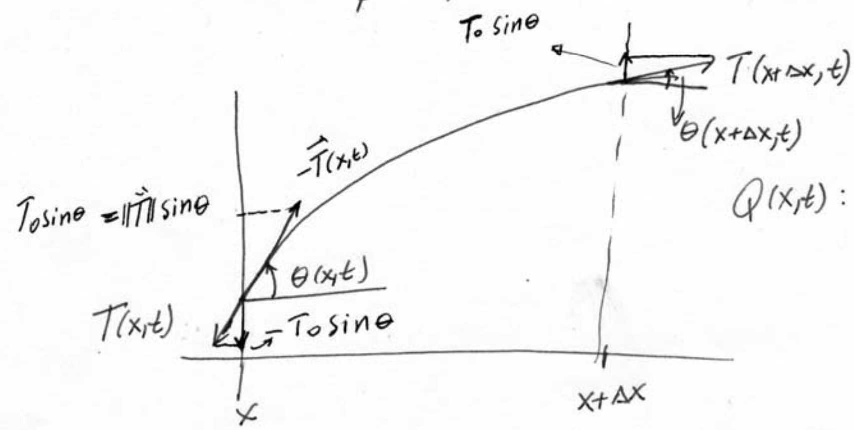
If Slope of String in motion is small v (horiz. displac.)
Can be neglected.

$y = u(x,t)$: Vertical displacement.

$\rho_0(x)$: mass density (linear).

Newton law for an infinitesimal portion of the string

$\vec{F} = \text{mass} \times \vec{a}$. $T(x,t)$: magnitude of tensile force
tangential force



$Q(x,t)$: Vertical component of body force/mass.

Newton Law Applied to an infinitesimal portion of the string.
$[x, x+\Delta x]$

Newton:

$$\rho_0(\bar{x})\,\Delta x\,\frac{\partial^2 u}{\partial t^2}(\bar{x},t) = T(x+\Delta x,t)\,\sin[\theta(x+\Delta x,t)] - $$
$$- T(x,t)\,\sin[\theta(x,t)] + \rho_0(\bar{x})\,\Delta x\, Q(\bar{x},t)$$

$$x \le \bar{x} \le x+\Delta x$$

div. by Δx and $\Delta x \to 0$. $\Rightarrow \bar{x} \to x$.

$$\boxed{\rho_0(x)\,\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left[T(x,t)\,\sin\theta(x,t)\right] + \rho_0(x)\,Q(x,t).} \quad (2.1)$$

Now, Slope at (x,t) is

$$\tan\theta(x,t) = \frac{\partial u}{\partial x}(x,t) \quad \text{(ignoring horiz. displac.)}$$

and for small angles θ,

$$\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \simeq \sin\theta$$

then (2.1) reduces to

$$\rho_0(x)\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x,t)\,\frac{\partial u}{\partial x}(x,t)\right] + \rho_0(x)\,Q(x,t)$$

perfectly elastic strings: $T(x,t) \simeq T_0$ because θ is small

Tension perturbed string almost the same as unperturbed string.

$$\Rightarrow \boxed{\rho_0(x)\,\frac{\partial^2 u}{\partial t^2} = T_0\,\frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho_0(x)}$$

If $Q(x,t)=0$, then

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}, \qquad c^2 \equiv T_0/\rho_0(x).$$

wave speed

If string is uniform then c is constant.

One-dimensional wave equation.

4.3

Boundary conditions:

a) Fixed ends: $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0. \end{cases}$

b) Prescribed Variations: $\begin{cases} u(0,t) = f(t) \\ u(L,t) = g(t) \end{cases}$

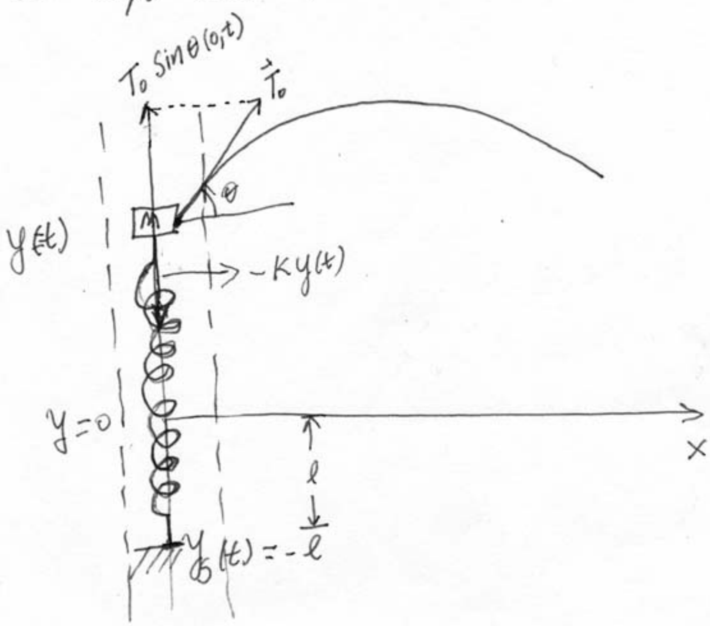
Initial conditions:

$u(x,0) = \phi(x)$

$u_t(x,0) = \psi(x)$

Why two IC's?

c) Left end $x=0$ is attached to a Spring-mass system.

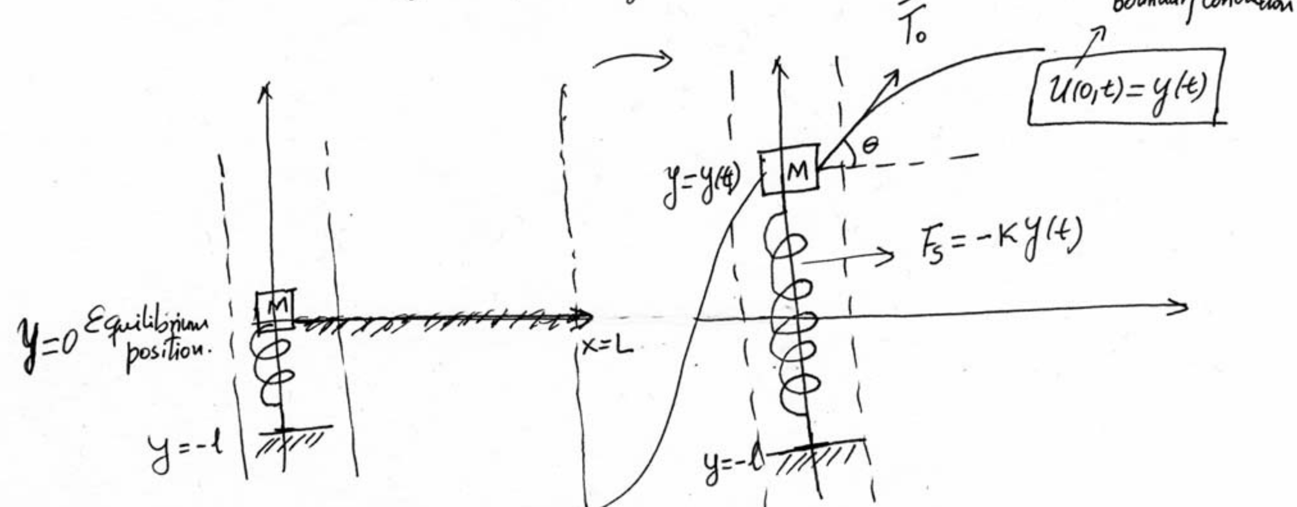


l: Spring natural length.

$y(t) = u(0,t)$. B.C.

c) <u>Boundary Conditions:</u> End of string at $x=0$ is attached to a Spring-mass system.

Support of the spring is located at $y=-l$. and length of spring (Natural) is l.



This mass is subject to two forces:

$$\left.\begin{array}{l} \uparrow T_0 \sin\theta(0,t) \simeq T_0 \dfrac{\partial u}{\partial x}(0,t) \\ \boxed{M} \\ \downarrow -Ky(t) = -K\,u(0,t) \end{array}\right\} \Rightarrow m\frac{d^2y}{dt^2} = -Ky(t) + T_0\frac{\partial u}{\partial x}(0,t)$$

<u>Equation at border:</u>

Assuming $m \ll 1$ (m very small) forces are in balance

$$\boxed{T_0\frac{\partial u}{\partial x}(0,t) = Ky(t) = K\,u(0,t)}$$

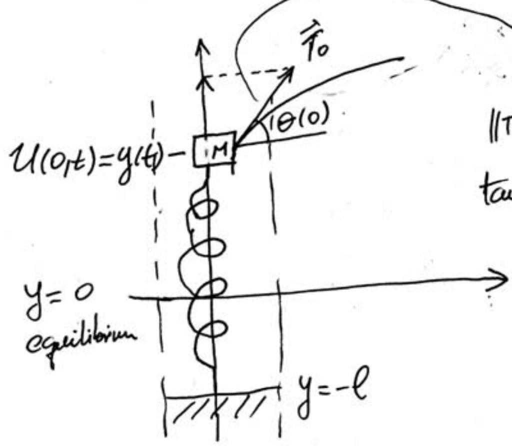
Elastic Bondary at $x=0$ Condition.

Diagram of the string attached to the mass-spring system when

a) Mass is located above the equilibrium position

b) " " " below the equilibrium position

a) Mass is above the equilibrium position



$\frac{\partial u}{\partial x}(0,t)>0$

$\|T_0\|\sin\theta(0) \simeq \|\vec{T}_0\| \frac{\partial u}{\partial x}(0,t) > 0$

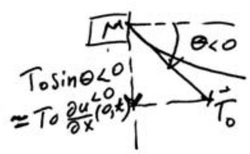
$\tan(\theta(0)) = \frac{\partial u}{\partial x}(0,t)$

$-K\overset{>0}{u(0,t)} < 0$

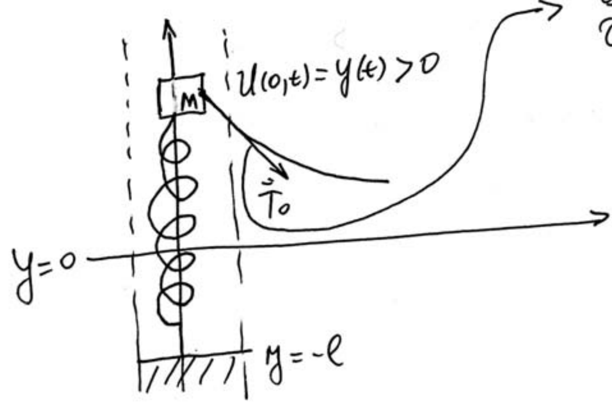
In balance.

$$\Rightarrow \boxed{T_0 \frac{\partial u}{\partial x}(0,t) = K\, u(0,t)}$$

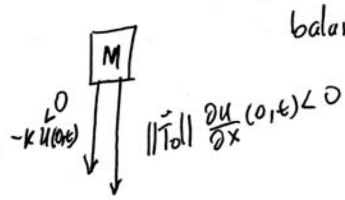
Another possible diagram:



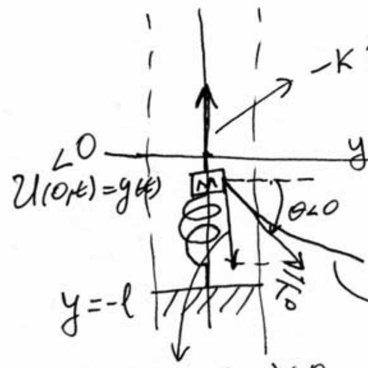
$\frac{\partial u}{\partial x}(0,t) < 0$



Not in balance.



b) Mass is located below the equilibrium position.



$\frac{\partial u}{\partial x}(0,t) < 0$

$U(0,t) < 0$

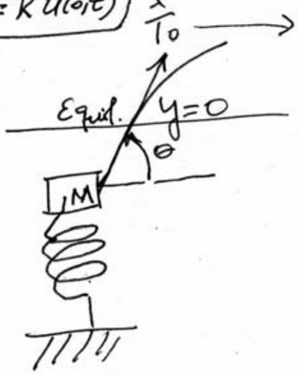
$\Rightarrow$

In balance

Therefore, for force balance:

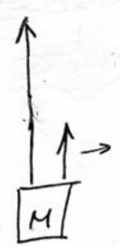
$$T_0 \frac{\partial u}{\partial x}(0,t) = K\,U(0,t)$$

The other possibility



$T_0 \sin\theta \simeq T_0 \tan\theta$

$= T_0 \frac{\partial u}{\partial x}(0,t) > 0$



Not in balance.

$$T_0 \sin\theta \simeq T_0 \frac{\partial u}{\partial x}(0,t)$$

A similar analysis at the other end

$$\boxed{T_0 \frac{\partial u}{\partial x}(L,t) = -k\, u(L,t).}$$

If $\underline{\underline{k=0}}$ $\quad T_0 \frac{\partial u}{\partial x}(0,t) = 0 \quad$ or $\quad T_0 \frac{\partial u}{\partial x}(L,t) = 0.$

Free end B.C.
or Tensile force vanishes at the end.

Separation of Variables

$$\begin{cases} u_{tt} = c^2 u_{xx}, \qquad 0<x<L. \\ \text{B.C.}\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}, \qquad \text{I.C's:}\begin{cases} u(x,0)=f(x) \\ u_t(x,0)=g(x). \end{cases} \end{cases}$$

<u>IBVP</u>

$$u(x,t) = \phi(x)\, h(t).$$

$$\phi(x)\, h''(t) = c^2 h(t)\, \phi''(x).$$

Dividing by $c^2\phi(x)h(t)$:

$$\frac{1}{c^2}\frac{h''(t)}{h(t)} = \frac{1}{\phi(x)}\phi''(x) = -\lambda.$$

Eigenvalue problem:

$$\phi''(x) + \lambda\phi(x) = 0$$
$$\phi(0) = 0, \quad \phi(L) = 0.$$

We have shown before that

$\underline{\underline{\lambda > 0}}$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

And $\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$.

ODE on time: $h''(t) + \left(\frac{n\pi}{L}\right)^2 c^2 h(t) = 0$

$$\Rightarrow \quad h(t) = C_1 \cos\frac{n\pi c}{L}t + C_2 \sin\frac{n\pi c}{L}t$$

Principle of Superposition:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)\right]$$

Assuming a term-term differentiation with respect to time is allowed

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}\left[-A_n \sin\left(\frac{n\pi}{L}ct\right)\sin\frac{n\pi}{L}x + B_n \cos\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)\right]$$

Then using I.C's.

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right).$$

and

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi}{L}x\right).$$

$$\Rightarrow \quad A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \qquad (9.1)$$

$$B_n \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx \qquad (9.2)$$

Again, the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi c}{L}t + B_n \sin \frac{n\pi}{L}ct\right] \sin\left(\frac{n\pi}{L}x\right) \qquad (9.3)$$

where A_n and B_n $n=1,2,\ldots$ are given by (9.1) and (9.2).

Physical Interpretation:

Normal modes of Vibration:

Vertical displacement is linear combination of product solutions:

$$\left[A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right)\right] \sin\left(\frac{n\pi}{L}x\right)$$

or normal modes of Vibration.

The expresion in bracket can be written as $\left(\omega \equiv \frac{n\pi}{L}c\right)$

$$A_n \cos(\omega t) + B_n \sin(\omega t) = R_n \sin(\omega t + \theta_n)$$

To find R_n and θ_n, we expand the right hand side.

$$A_n \cos(\omega t) + B_n \sin(\omega t) = R_n\left[\sin(\omega t)\cos\theta_n + \cos(\omega t)\sin\theta_n\right]$$

$$= \left(R_n \cos\theta_n\right) \sin(\omega t) + \left(R_n \sin\theta_n\right) \cos\omega t$$

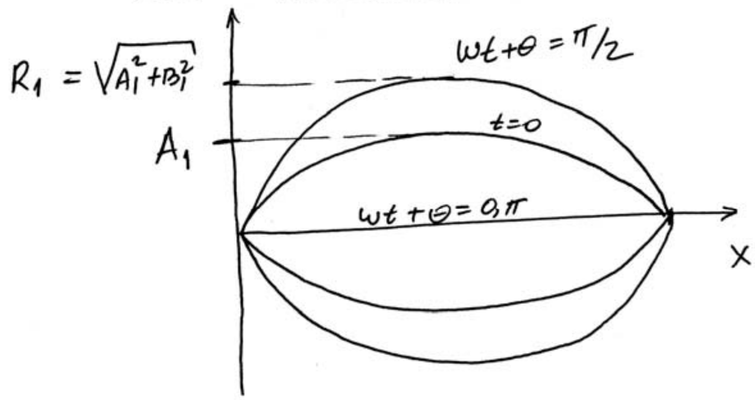
$$\Rightarrow \quad R_n \cos\theta_n = B_n \quad , \quad R_n \sin\theta_n = A_n$$

$$\Rightarrow \quad \frac{\cancel{R} \sin\theta_n}{\cancel{R} \cos\theta_n} = \tan\theta_n = \frac{A_n}{B_n} \Rightarrow \boxed{\theta_n = \tan^{-1}\left(\frac{A_n}{B_n}\right)}$$

Also, $\quad A_n^2 + B_n^2 = R_n^2 \sin^2\theta_n + R^2 \cos^2\theta_n = R^2$

$$\Rightarrow \quad \boxed{R_n = \sqrt{A_n^2 + B_n^2}.}$$

$\underline{n=1}$ Fundamental Mode



$R_1 = \sqrt{A_1^2+B_1^2}$

$U_1(x,t) = R_1 \sin\left(\frac{\pi c}{L} t + \theta\right)$

If $t=0$

$U_1(x,t) = R_1 \sin(\theta)$

$= A_1$

If $\omega t + \theta = \pi/2$

$U_1(x,t) = R_1 \sin(\pi/2) = R_1 = \sqrt{A_1^2+B_1^2}$.

$U_1(x,t) = A_1 \cos \omega t + B_1 \sin \omega t = R_1 \sin(\omega t + \theta)$.

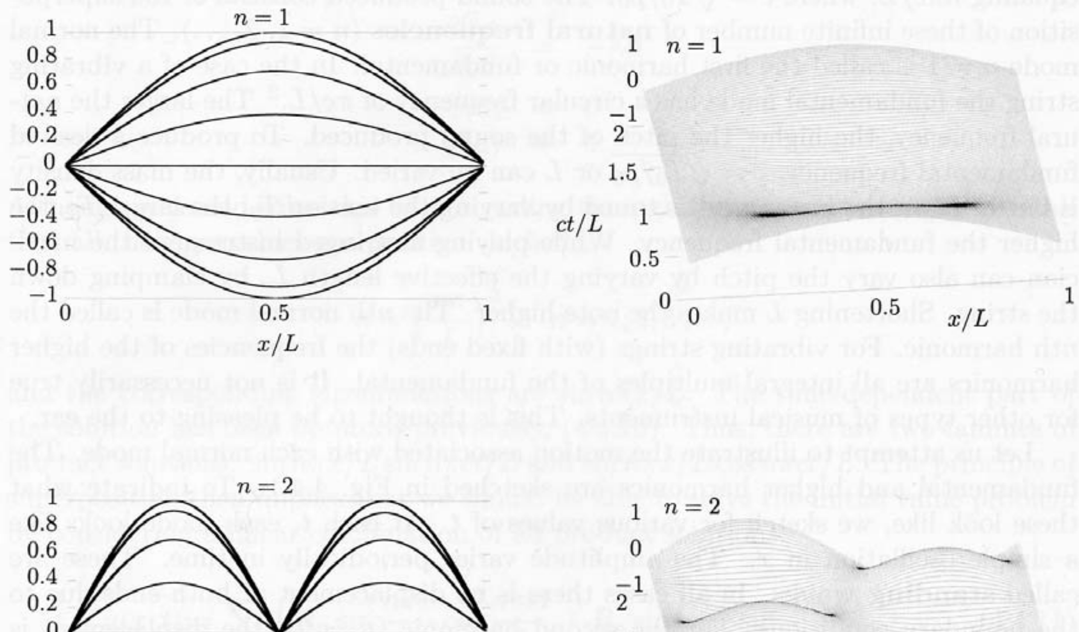


Figure 4.4.1 Normal modes of vibration for a string.

Each mode is simple harmonic in time

Circular frequency or Natural frequency : $\frac{n\pi c}{L}$: # of oscilations in 2π units of time.

$\underline{\underline{n=1}}$ Harmonic or Fundamental mode

Circular or natural freq $= \frac{\pi c}{L}$ (oscilations in 2π units of time)

oscilation/unit time $= \frac{\pi c/L}{2\pi} = \frac{c}{2L}$ Cycles/time.

Motion of each mode:

For t fixed each mode is a simple oscilation in x.

the amplitude: $R_n \sin(\omega t + \theta_n)$ Varies periodically in time.

they are called standing waves

2nd harmonic: Displac. zero at $x = L/2$ for all time t.

$x = L/2$ is called a node.

The n^{th} mode has $(n-1)$ nodes.

Every term in each mode is composed of two traveling waves. In fact,

$$\boxed{\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi c}{L}t\right) = \underbrace{\frac{1}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right)}_{\substack{\text{Wave traveling}\\ \text{to the right}}} - \underbrace{\frac{1}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right)}_{\substack{\text{Wave traveling}\\ \text{to the left.}}}}$$

Since

$$\frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] =$$

$$\frac{1}{2}\left[\cancel{\cos\alpha\cos\beta} + \sin\alpha\sin\beta - \cancel{\cos\alpha\cos\beta} + \sin\alpha\sin\beta\right] = \underline{\sin\alpha\sin\beta}.$$

Therefore, solution of wave equation can be written as.

$$\boxed{u(x,t) = R(x-ct) + S(x+ct).}$$

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t>0, \quad 0<x<L \\ u(0,t)=0, \quad u(L,t)=0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{L} ct\right) + B_n \sin\left(\frac{n\pi}{L} ct\right)\right] \sin \frac{n\pi}{L} x$$

Solution obtained using separation of variables.

where

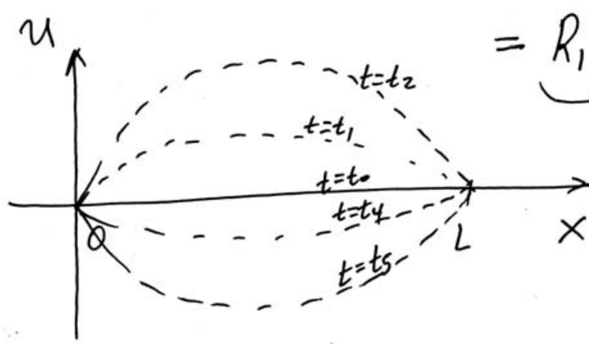
$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx.$$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \quad dx$$

Now $\quad C = \sqrt{\frac{T_0}{\rho_0}}$

For $n=1$: $\left[A_1 \cos\left(\frac{\pi}{L} ct\right) + B_1 \sin\left(\frac{\pi}{L} ct\right)\right] \sin \frac{\pi}{L} x.$

$$= \underbrace{R_1 \sin\left(\frac{\pi}{L} ct + \theta\right)}_{\text{Amplitude at time } t.} \sin\left(\frac{\pi}{L} x\right)$$



$R_1 = \sqrt{A_1^2 + B_1^2}$