3.4 Accuracy and Stability

The 5 point stencil for a 2-D Poisson boundary value problem consists of the equations:

$$\frac{1}{h^2}\left[U_{i,j-1} + U_{i-1,j} - 4U_{ij} + U_{i+1,j} + U_{i,j+1}\right] = f_{ij} \qquad (1.1)$$

$\left.\begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, m. \end{array}\right\}$ interior points

This is assuming:

a) Dirichlet boundary condition at the boundary $\partial\Omega$.

b) Boundary $\partial\Omega$ is a Square region. Otherwise, (1.1) is still valid, but we need special treatment at points on $\partial\Omega$.

c) Number of grid points $m+2$ points is the same in both directions.

Recall that the more general case when

- $\Delta x \neq \Delta y$
- $\partial\Omega$ rectangular region (not necessarily square)
- Not same number of points in both direction

$$\theta_y U_{i,j-1} + \theta_x U_{i-1,j} - U_{ij} + \theta_x U_{i+1,j} + \theta_y U_{i,j+1} = \theta_{xy} f_{ij}$$

where $\theta_x = \dfrac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}$, $\theta_y = \dfrac{\Delta x^2}{2(\Delta x^2+\Delta y^2)}$ $\quad i = 1, \dots, m_x$ $\quad$ (1.2)

$j = 1, \dots, m_y$

$$\theta_{xy} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

The more general stencil (1.2) reduces to (1.1) when $\Delta x = \Delta y$ (Verify this).

the linear system obtained from (1.1) is

$$\boxed{A\vec{U} = \vec{F}} \qquad (2.1)$$

where (for row-ordering)

$$A = \frac{1}{h^2}\begin{bmatrix} T & I & 0 & 0 & \cdots & 0 \\ I & T & I & & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & I \\ 0 & \cdots & & 0 & I & T \end{bmatrix}_{m^2\times m^2}, \quad T \equiv \begin{bmatrix} -4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & 1 \\ 0 & \cdots & & & 1 & -4 \end{bmatrix}_{m\times m}$$

For Dirichlet condition $\boxed{u(x,y) = 0,}$ $(x,y) \in \partial\Omega$

$$I = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \ddots & \\ \cdots & 0 & 1 \end{bmatrix}_{m\times m}$$

$$\vec{U} = \begin{bmatrix} U_{11} \\ U_{21} \\ \vdots \\ U_{m1} \\ U_{12} \\ U_{22} \\ \vdots \\ U_{m2} \\ \vdots \\ U_{1m} \\ U_{2m} \\ \vdots \\ U_{mm} \end{bmatrix}_{m^2\times 1}, \qquad \vec{F} = \begin{bmatrix} f_{11} \\ f_{21} \\ \vdots \\ f_{m1} \\ f_{12} \\ \vdots \\ f_{m2} \\ \vdots \\ f_{1m} \\ \vdots \\ f_{mm} \end{bmatrix}_{m^2\times 1}$$

The local truncation error τ_{ij} of (1.1) is obtained by substituting the soln. $u(x,y)$ of

$$\nabla^2 u = f$$

into (1.1) and by splitting it into the second order centered differences in x and y

In fact,

$$\tau_{ij} = \frac{1}{h^2}\Big[\underbrace{u_{i-1,j} - 2u_{ij} + u_{i+1,j}} + \underbrace{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}\Big] - f_{ij}$$

$$= \frac{h^2}{12}\left[u_{xxxx} + u_{yyyy}\right] + O(h^4).$$

Defining global error

$$\vec{E} = \vec{U} - \vec{u}$$

we obtain

$$A\vec{E} = A\vec{U} - A\vec{u} = \vec{F} - (\vec{F} + \vec{\tau}) = -\vec{\tau}$$

The method will be of second order (globally) if it is stable in some norm, equivalent to say

$$\|(A^h)^{-1}\| \leq C, \text{ for all } h < h_0$$

Substituting the exact solution u_{ij} into (1.1)

$$\frac{1}{h^2}\left(u_{i,j-1} - 2u_{ij} + u_{i,j+1} + u_{i-1,j} - 2u_{ij} + u_{i+1,j}\right) - f_{ij}$$

$$= (u_{xx})_{ij} + \frac{h^2}{12}(u_{4x})_{ij} + (u_{yy})_{ij} + \frac{h^2}{12}(u_{4y})_{ij} + O(h^4) - f_{ij}$$

$$= \underbrace{\cancel{(\nabla^2 u)_{ij}}}_{= f_{ij}} + \frac{h^2}{12}\left[u_{4x} + u_{4y}\right]_{ij} + O(h^4) \cancel{- f_{ij}}$$

$$= \frac{h^2}{12}\left[u_{4x} + u_{4y}\right]_{ij} + O(h^4) = \tau_{ij}$$

Therefore,

$$\frac{1}{h^2}\left[u_{i,j-1} - 2u_{ij} + u_{i,j+1} + u_{i-1,j} - 2u_{ij} + u_{i+1,j}\right]$$

$$= f_{ij} + \tau_{ij}$$

and $A\vec{u} = \vec{F} + \vec{\tau}$, where

where $\vec{F} = \begin{bmatrix} f_{11} \\ f_{21} \\ \vdots \\ f_{m1} \\ f_{12} \\ \vdots \\ f_{m2} \\ \vdots \\ f_{1m} \\ f_{2m} \\ \vdots \\ f_{mm} \end{bmatrix}$, $\vec{\tau} = \begin{bmatrix} \tau_{11} \\ \tau_{21} \\ \vdots \\ \tau_{m1} \\ \vdots \\ \vdots \\ \vdots \\ \tau_{1m} \\ \tau_{2m} \\ \vdots \\ \tau_{mm} \end{bmatrix}$

In this case, it is convenient to use the 2-norm.

The eigenvalues of A can be computed explicitly.

<u>Eigenvectors:</u>

$$\boxed{u_{ij}^{p,q} = \sin(p\pi i h)\sin(q\pi j h)} \quad (4.1)$$

$$p,q = 1,..,m \quad , \qquad i,j = 1,..,m.$$

<u>Eigenvalues:</u>

$$\boxed{\lambda_{p,q} = \frac{2}{h^2}\left[\left(\cos(p\pi h)-1\right)+\left(\cos(q\pi h)-1\right)\right]} \quad (4.2)$$

This result is expected if we consider the continuous case:

<u>Eigenvalue Problem:</u>

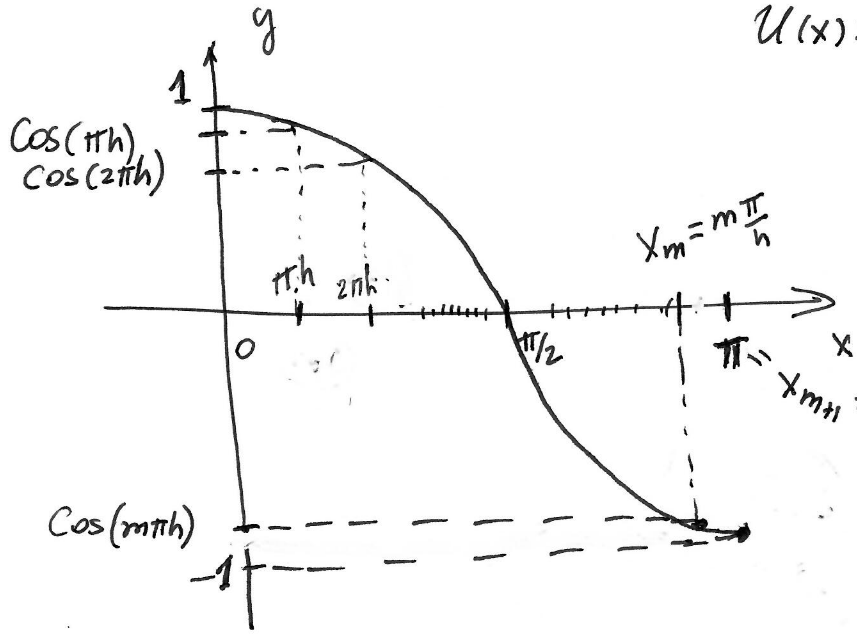
$$\begin{cases} \nabla^2 u = u_{xx}+u_{yy} = \lambda u \,, & 0<x,y<1. \\ u(x,y)=0, & (x,y)\in \text{Bdry} \ \square \end{cases} \quad (4.3)$$

$$\boxed{\text{Eigenvalues: } \lambda_{p,q} = -(p\pi)^2-(q\pi)^2}$$

$$\boxed{\text{Eigenvectors: } u_{pq}(x,y) = \sin(p\pi x)\sin(q\pi y).} \quad (4.4)$$

We can easily verified that (4.3) and (4.4) are obtained from (4.1) and (4.2) when $h \to 0$.

Consider



$u(x) = \cos x$

$h = \frac{1}{m+1}$

$\Rightarrow m+1 = \frac{1}{h}$

Finding out the smallest and larger eigenvalues $\lambda_{p,q}$

$p,q = 1,2,.., m$

Notice that

$$0 \leq \pi h \leq p\pi h \leq m\pi h \leq (m+1)\pi h = \pi$$

$$\Rightarrow \quad |\cos(\pi h) - 1| \leq |\cos(p\pi h) - 1| \leq |\cos(\pi) - 1| = 2$$

then $|\lambda_1| = |\cos(\pi h) - 1|$ smallest eigenvalue in magnitude

$|\lambda_m| = |\cos(m\pi h) - 1|$ largest. " " ".

Back to 2D-case.

$$u_{ij}^{p,q} = \sin(p\pi i h)\sin(q\pi j h)$$

$$\lambda_{p,q} = \frac{2}{h^2}\left[\cos(p\pi h - 1) + \cos(q\pi h - 1)\right]$$

$p, q = 1, .., m$
$i, j = 1, .., m.$

Similar to the 1-D case

$$\lambda_{p,q} = \frac{2}{h^2}\Big[\cancel{1} - \frac{1}{2}p^2\pi^2h^2 + \frac{1}{4!}p^4\pi^4h^4 + \cdots - \cancel{1}$$

$$+ \cancel{1} - \frac{1}{2}q^2\pi^2h^2 + \frac{1}{4!}q^4\pi^4h^4 + \cdots - \cancel{1}\Big]$$

$$= -(p^2+q^2)\pi^2 + \mathcal{O}(h^2).$$

So for $h \ll 1$

$$|\lambda_{p,q}| \simeq (p^2+q^2)\pi^2$$

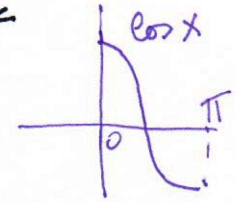
and $\lambda_{1,1}$ is the eigenvalue with smallest magnitude.

and $\lambda_{m,m}$ is " " " largest magnitude.

therefore $\|(A^h)^{-1}\| = \left(\min_{1\le p,q\le m} \lambda_{p,q}\right)^{-1} = \frac{1}{2\pi^2} \le C.$

As a consequence, the method is stable and it is also convergent of $\mathcal{O}(h^2)$.

For the matrix A given by (1.1) or (3.12) in book



$$\boxed{U_{ij}^{p,q} = \sin(p\pi i h)\sin(q\pi j h)} \quad \text{eigenvects}$$

$$\boxed{\lambda_{p,q} = \frac{2}{h^2}\left[(\cos(p\pi h)-1)+(\cos(q\pi h)-1)\right] \quad \text{Eigenvalues} \quad p,q=1,2\ldots m}$$

Eigenvs. strictly negative $\Rightarrow$ A is negative definite $\quad h=\frac{1}{m+1}$

Eigenv. closest to the origin for any h

$$\boxed{\lambda_{1,1} = -2\pi^2 + \mathcal{O}(h^2)} \qquad A^h \text{ is symmetric.}$$

$$\Rightarrow \|(A^h)^{-1}\|_2 = \rho((A^h)^{-1}) = \frac{1}{|\lambda_{1,1}|} \simeq +\frac{1}{2\pi^2} \Rightarrow \text{FDM is stable in } \|\cdot\|_2.$$

Comment on Cond. # = $\boxed{K_2(A) = \|A\|_2\|A^{-1}\|_2}$

Largest eigenv. of A = $\boxed{\lambda_{m,m} \simeq -\frac{\cancel{1}8}{h^2} \Rightarrow \|A^h\|_2 = \frac{8}{h^2}}$

$$\Rightarrow \boxed{K_2(A^h) \simeq \frac{4}{\pi^2 h^2} = \mathcal{O}\left(\frac{1}{h^2}\right) \xrightarrow[\text{as } h\to 0]{} \infty}$$

Very bad conditioned!

$$\lambda_{m,m} = -2m^2\pi^2 \simeq -2\left(\frac{1}{h}\right)^2\pi^2 = -\frac{2\pi^2}{h^2} \simeq -\frac{18}{h^2}$$

Then $K_2(A^h) \simeq \|A\|_2\|A^{-1}\|_2 \simeq +\frac{18}{h^2}\left(+\frac{1}{2\pi^2}\right) \simeq \frac{9}{h^2\pi^2} = \mathcal{O}\left(\frac{1}{h^2}\right)$ $\xrightarrow{h\to 0} \infty$