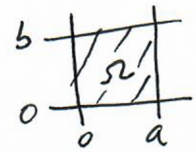
SOR Applied to Poisson's Equation.

BVP: $\begin{cases} \nabla^2 u = f(x,y), & (x,y) \in \Omega, \\ u(\vec{x}_s) = \gamma(\vec{x}_s), & \vec{x}_s \in \partial\Omega \end{cases}$



Using 5-point stencil finite difference method (2nd order centered fdm).

$$\boxed{\nabla_5^2 U_{jk} = f_{ij}} \qquad (0.1)$$

$$\boxed{\frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{(\Delta x)^2} + \frac{U_{j,k+1} - 2U_{jk} + U_{j,k-1}}{\Delta y^2} = f_{jk}}$$

In matrix form: $\boxed{A\vec{U} = \vec{f}}$

Solving for U_{jk}, $j = 1, \dots, N_x - 1$, $k = 1, \dots, N_y - 1$ (1.1)

$$\boxed{U_{jk} = \theta_x \left(U_{j+1,k} + U_{j-1,k} \right) + \theta_y \left(U_{j,k+1} + U_{j,k-1} \right) - \theta_f f_{jk}}$$

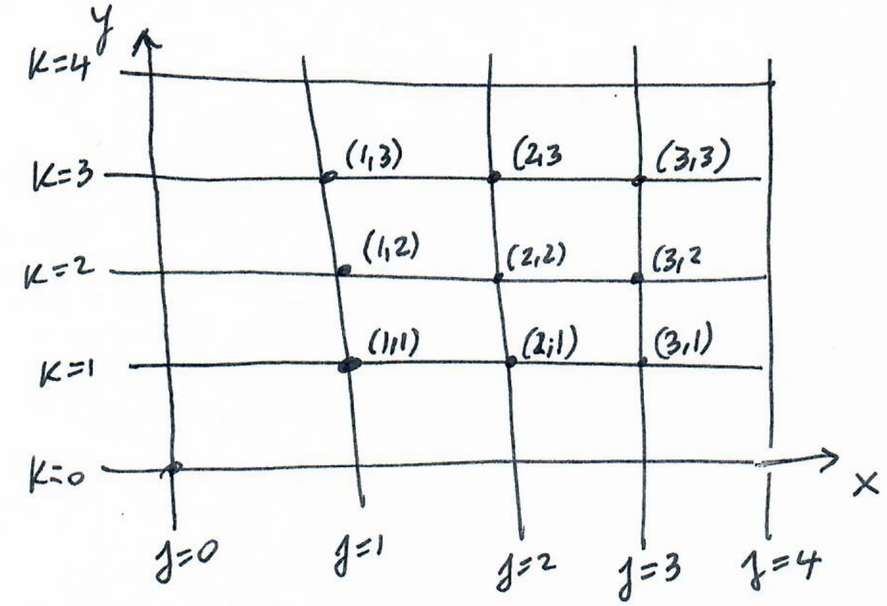
(1.2)

where $\theta_x = \dfrac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}$, $\theta_y = \dfrac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}$

and

$$\theta_f = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

For $N_x = 4$, $N_y = 4$ DIRICHLET PROBLEM



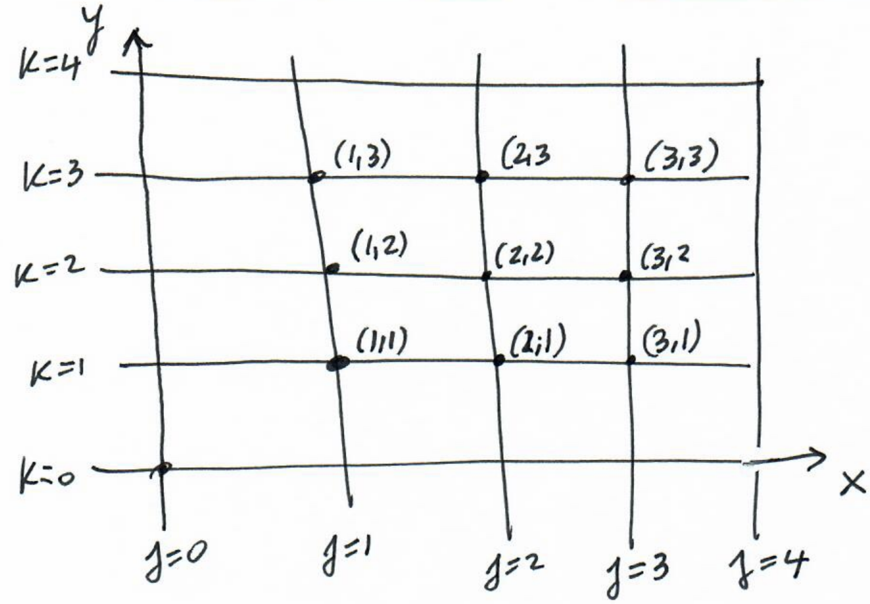
$$
A = \begin{array}{c} \\ K=1 \\ \\ \\ K=2 \\ \\ \\ K=3 \\ \\ \end{array}
\left(\begin{array}{ccc|ccc|ccc}
-1 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 & 0 \\
\theta_x & -1 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 \\
0 & \theta_x & -1 & 0 & 0 & \theta_y & 0 & 0 & 0 \\
\hline
\theta_y & 0 & 0 & -1 & \theta_x & 0 & \theta_y & 0 & 0 \\
0 & \theta_y & 0 & \theta_x & -1 & \theta_x & 0 & \theta_y & 0 \\
0 & 0 & \theta_y & 0 & \theta_x & -1 & 0 & 0 & \theta_y \\
\hline
0 & 0 & 0 & \theta_y & 0 & 0 & -1 & \theta_x & 0 \\
0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & -1 & \theta_x \\
0 & 0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & -1
\end{array}\right)
\begin{array}{c} j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \end{array}
\begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{12} \\ v_{22} \\ v_{32} \\ v_{13} \\ v_{23} \\ v_{33} \end{pmatrix}
$$

A $\qquad\qquad\qquad$ $\vec{v}$

Where

$$\vec{v} = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{N_x-1,1} \\ v_{12} \\ \vdots \\ v_{N_x-1,2} \\ \vdots \\ v_{1,N_y-1} \\ \vdots \\ v_{N_x-1,N_y-1} \end{bmatrix}, \qquad \vec{C} = -\theta_f \begin{bmatrix} f_{11} \\ \vdots \\ f_{N_x-1,1} \\ f_{12} \\ \vdots \\ f_{N_x-1,2} \\ \vdots \\ f_{1,N_y-1} \\ \vdots \\ f_{N_x-1,N_y-1} \end{bmatrix} \qquad (a.1)$$

For $N_x = 4, \ N_y = 4$ DIRICHLET PROBLEM



(10.1)

$$
M = \begin{array}{c} \\ k=1 \\ \\ \\ k=2 \\ \\ \\ k=3 \\ \\ \end{array}
\left(\begin{array}{ccc|ccc|ccc}
0 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 & 0 \\
\theta_x & 0 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 \\
0 & \theta_x & 0 & 0 & 0 & \theta_y & 0 & 0 & 0 \\
\hline
\theta_y & 0 & 0 & 0 & \theta_x & 0 & \theta_y & & 0 \\
0 & \theta_y & 0 & \theta_x & 0 & \theta_x & 0 & \theta_y & 0 \\
0 & 0 & \theta_y & 0 & \theta_x & 0 & 0 & 0 & \theta_y \\
\hline
0 & 0 & 0 & \theta_y & 0 & 0 & 0 & \theta_x & 0 \\
0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & 0 & \theta_x \\
0 & 0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & 0
\end{array}\right)
\begin{array}{l} j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \end{array}
\qquad (10.2)
$$

Iterative techniques:

$K = 1, \ldots, N_x - 1$

a) Jacobi: $j = 1, \ldots, N_y - 1$

$$\boxed{U_{jk}^{(l+1)} = \theta_x \left(U_{j+1,k}^{(l)} + U_{j-1,k}^{(l)} \right) + \theta_y \left(U_{j,k+1}^{(l)} + U_{j,k-1}^{(l)} \right) - \theta_f f_{jk}} \tag{5.1}$$

b) Gauss-Seidel:

$$\boxed{U_{jk}^{(l+1)} = \theta_x \left(U_{j+1,k}^{(l)} + U_{j-1,k}^{(l+1)} \right) + \theta_y \left(U_{j,k+1}^{(l)} + U_{j,k-1}^{(l+1)} \right) - \theta_f f_{jk}} \tag{5.2}$$

c) SOR:

$$\boxed{U_{jk}^{(l+1)} = \omega \left(U_{jk}^{(l+1)} \right)_{GS} + (1-\omega)\, U_{jk}^{(l)}} \tag{5.3}$$

In matrix form, as a fixed point problem:

$$\vec{U} = M_\omega \vec{U} + C_\omega$$

Key questions:
i) Conditions on original matrix A for converg.
ii) Appropriate range of ω for convergence.
iii) Finding optimum ω.
iv) Rate of convergence.

Summary

Finding the ω optimum ω_{OPT} for SOR applied to Poisson equation.

(I) Poisson Equ. (2D): $\boxed{u_{xx} + u_{yy} = f(x,y)}$

(II) 5-point scheme:

$$\boxed{\frac{v_{j,k+1} - 2v_{jk} + v_{j,k-1}}{(\Delta y)^2} + \frac{v_{j+1,k} - 2v_{jk} + v_{j-1,k}}{(\Delta x)^2} = f_{jk}}$$

(III) Direct Scheme:

$$\boxed{\theta_y v_{j,k-1} + \theta_x v_{k-1,j} - v_{jk} + \theta_x v_{k+1,j} + \theta_y v_{j,k+1} = \theta_f f_{jk}}$$

where

$$\theta_y = \frac{(\Delta x)^2}{2(\Delta x^2 + \Delta y^2)}, \quad \theta_x = \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}, \quad \theta_f = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

(IV) Fixed-Point Problem:

$$\boxed{v_{jk} = \theta_x \left(v_{j+1,k} + v_{j-1,k}\right) + \theta_y \left(v_{j,k+1} + v_{j,k-1}\right) - \theta_f f_{jk}}$$

or

$$\boxed{\vec{v} = M\vec{v} + \vec{C}}$$

where M is defined as follows assuming $N_x = 4 = N_y$.

Three Possible Iterative Techniques.

① $\vec{v}^{(l+1)} = M_J \vec{v}^{(l)} + \vec{C}$

② $\vec{v}^{(l+1)} = M_{gs} \vec{v}^{(l)} + \vec{C}_{gs}$

③ $\vec{v}^{(l+1)} = M_\omega \vec{v}^{(l)} + \vec{C}_s$

(Kaham)

(V) <u>Range of ω</u>: If $a_{ii} \neq 0$ (which is our case $a_{ii} = 1$)
Poisson's Equ.

then If SOR converges $\Rightarrow$ $0 < \omega < 2$.

(VI) <u>Convergence SOR:</u>

$\boxed{\vec{v}^{(k)} = M_\omega \vec{v}^{(k-1)} + C_\omega}$ converges to $\vec{v}^*$

fixed point of $\vec{v} = M_\omega \vec{v} + \vec{C}$. for $0 < \omega < 2$

and for any initial Guess.

(VII) <u>ω optimum of SOR and spectral radius $\rho(M_{\omega_{opt}})$.</u>

$$\boxed{\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2(M_j)}}}$$

and $\boxed{\rho(M_{\omega_{opt}}) = \omega_{opt} - 1}$

(VIII) <u>Comparison of spectral radius</u>

$\boxed{\rho(M_j) \simeq 1 - \frac{\pi^2}{2N_x^2}}$, $\boxed{\rho(M_g) \simeq 1 - \frac{\pi^2}{N_x^2}}$

$\boxed{\rho(M_{\omega_{opt}}) \simeq 1 - \frac{\pi}{N_x}}$

Computation of $\rho(M_j)$

IX Eigenvalues of M_j

$$\boxed{\mu_{mn} = 1 - 4\theta_x \sin^2\left(\frac{m\pi}{2N_x}\right) - 4\theta_y \sin^2\left(\frac{n\pi}{2N_y}\right)}$$

$m = 1, \ldots, N_x - 1$
$n = 1, \ldots, N_y - 1$

If $N_x = N_y$, $\Delta x = \Delta y$.

$$\boxed{\mu_{mn} = 1 - \sin^2\left(\frac{m\pi}{2N_x}\right) - \sin^2\left(\frac{n\pi}{2N_x}\right)}$$

X Largest Eigenvalue: When $m = n = 1$

$$\boxed{\mu_{1,1} = 1 - 2\sin^2\left(\frac{\pi}{2N_x}\right)}$$

W_{opt} when $\Delta x = \Delta y$ but $N_x \neq N_y$

$$\boxed{\rho(M_j) = 1 - \sin^2\frac{\pi}{2N_x} - \sin^2\frac{\pi}{2N_y} = \frac{1}{2}\left(\cos\frac{\pi}{N_x} + \cos\frac{\pi}{N_y}\right)}$$

and

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho^2(M_J)}} = \frac{2}{1+\sqrt{1-\frac{1}{4}\left(\cos\frac{\pi}{N_x}+\cos\frac{\pi}{N_y}\right)^2}}$$

or

$$\boxed{\omega_{opt} = \frac{4}{2+\sqrt{4-\left(\cos\frac{n\pi}{N_x}+\cos\left(\frac{\pi}{N_y}\right)\right)^2 d_s}}}$$