# 2.17 Singular Perturbation and Boundary Layers. Non-uniform Grids

Consider advection – diffusion equation.

$$\begin{cases} u_t + a u_x = \kappa u_{xx} + \psi, & a>0 \quad (1) \\ u(0,t) = \alpha(t) & 0<x<1, \ \kappa>0, \ t>0 \\ u(1,t) = \beta(t) \\ u(x,0) = h(x) \end{cases}$$

Discuss BCs.

i) For $a>0$, natural to have $u(0,t) = \alpha(t)$

ii) If $\kappa>0$, also need $u(1,t) = \beta(t)$ (?)

Steady-state

$$\begin{cases} a u'(x) = \kappa u''(x) + \psi(x) \\ u(0)=\alpha, \quad u(1)=\beta \end{cases} \qquad (2)$$

Cases:

i) $a=0 \Rightarrow$ steady-state heat conduction.

ii) $a \ll \kappa$ no problem!

iii) $a \gg \kappa$, $a>0 \Rightarrow$ boundary layer at $x=1$ end.

In case (iii), dividing by $a$

$$\frac{\kappa}{a} u''(x) - u'(x) = -\frac{\psi(x)}{a} \equiv f(x) \qquad \frac{1}{\varepsilon} = \frac{a}{\kappa} : \text{Péclet Number}$$

or

$$\boxed{\varepsilon u''(x) - u'(x) = f(x)} \qquad \text{Singular Perturbation Problem.} \quad (3)$$

$$\boxed{u(0)=\alpha, \quad u(1)=\beta}$$

Problems when $\varepsilon \to 0$, because higher order derivative term in (3) is almost zero. If $\varepsilon = 0$, then (3) reduces to

$$-u'(x) = f(x) \qquad (4)$$

It only needs one BC. Also, if $f(x) \equiv -1$.
The Soln. for BVP (3) is given by

$$u(x) = \alpha + x + (\beta - \alpha - 1)\left[\frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}\right]$$

Some Solns. for different $\varepsilon$ and $\boxed{\alpha = 1, \ \beta = 3}$ and $f(x) \equiv -1$ are shown in Fig. 2.6

It's seen that for small $\varepsilon$, the Soln. exhibits a high gradient close to the end $x=1$. It jumps from 2 to 3 close to $x=1$. when $\varepsilon \to 0$. This region of rapid changes is called boundary layer.
The soln. of (4) with B.C. at the left $u(0) = \alpha$, $f(x) = -1$ is

$$\boxed{u(x) = x + \alpha}$$

$$\lim_{x \to 1} u(x) = 1 + \alpha = 1 + 1 = 2$$

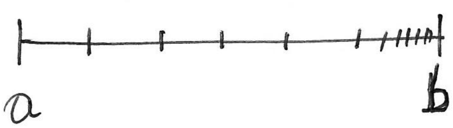
Since derivatives close to $x=1$ are large then F.D. methods produce large errors.

Fig 2.6 Suggests the use of nonuniform grids.

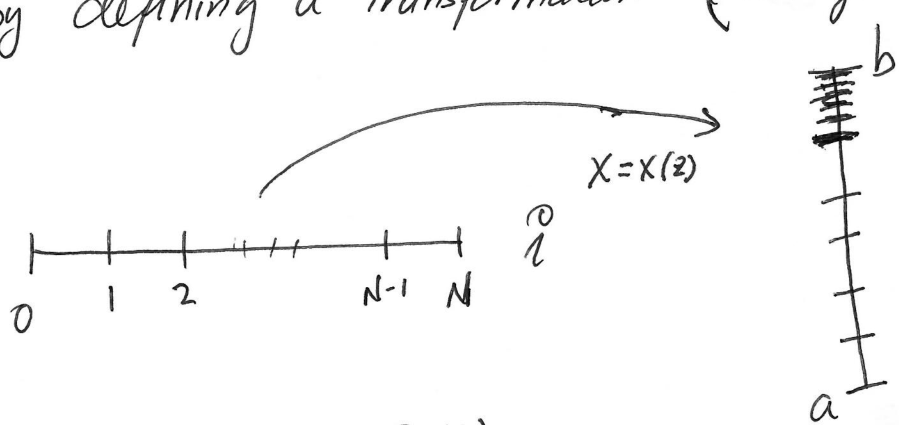
# Nonuniform Grids

From our previous example, we identify the need to construct grids with cluster of points in some regions of the domain. It depends on the behavior of the soln of the BVP in the neighborhood of certain points.

So, our goal is to define a grid as




A way to do that in a "smooth way" is by defining a transformation (using differentiable functions)

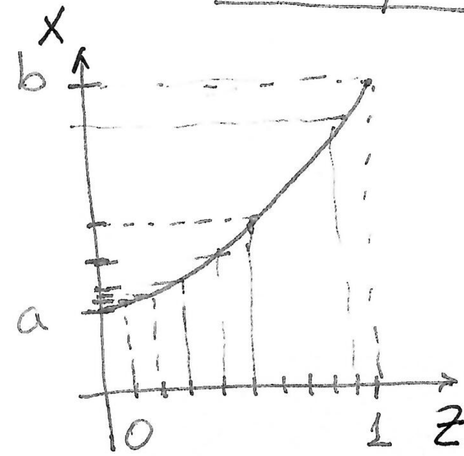


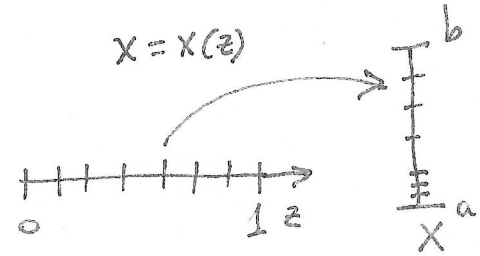

$$Z_i = Z(i)$$

$$i = 0, 1, \dots, N.$$

$$X_i = X(Z(i))$$

## Nonuniform grid with a cluster of points at the left end.




$$\boxed{X = X(z) = a + (b-a)z^2} \qquad 0 \le z \le 1$$

In fact,

$$X(z) = C_1 + C_2 z^2$$

$$X(0) = a, \quad X(1) = b$$

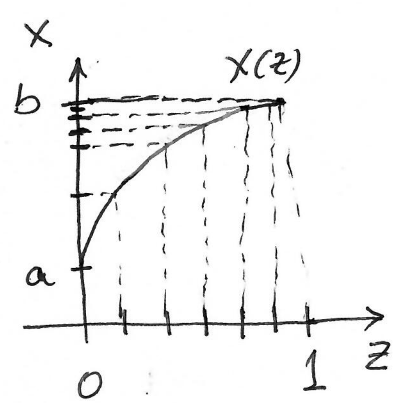
$$\boxed{C_1 = a}$$

$$C_1 + C_2 = b \Rightarrow \boxed{C_2 = b - C_1 = b - a}$$

$$\Rightarrow \boxed{X(z) = a + (b-a)z^2}$$

## Nonuniform grid with a cluster of points at the right end.

Clearly, a parabola like this would work to get a cluster of points at the upper end $b$.



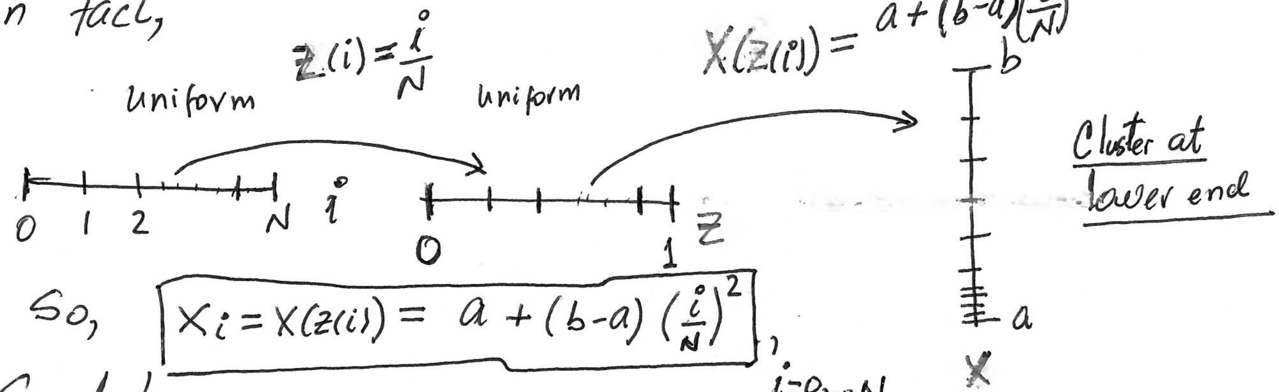

$$X(z) = C(x-1)^2 + d$$

Using B.C.: $X(1) = b$, $X(0) = a$

$$\left.\begin{aligned} X(0) &= C + d = a \\ X(1) &= d = b \end{aligned}\right\} \Rightarrow \boxed{X(z) = (a-b)(z-1)^2 + b}$$

These two transformations generate clustered grids of N points at the lower and upper end, respectively.

In fact,




So, $$x_i = X(z(i)) = a + (b-a)\left(\frac{i}{N}\right)^2, \quad i = 0, \dots, N$$

Similarly,

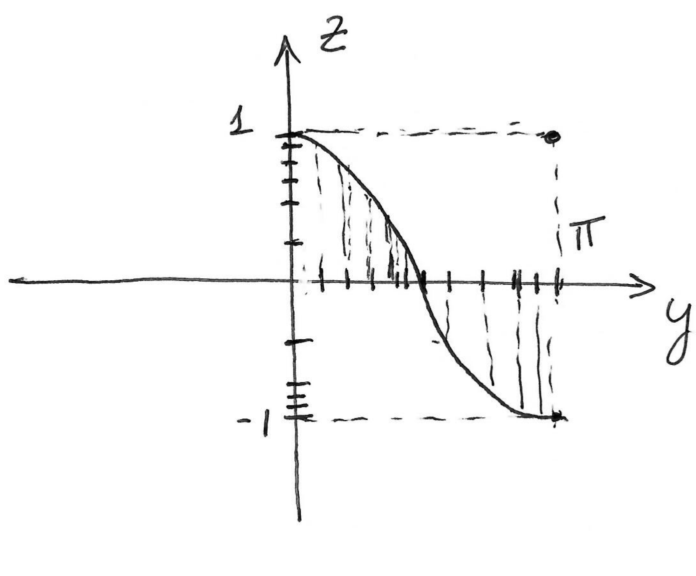
$$X_i = X(z(i)) = (a-b)\left(\frac{i}{N} - 1\right)^2 + b$$

will generate a clustered grid at the upper end.

## Cluster of Grid points at both ends Simultaneously

Chebyshev nodes:

Consider $z(y) = \cos(y), \quad 0 \le y \le \pi$




So, $z(y)$ transforms a uniform grid in $[0, \pi]$ into a nonuniform grid in $[-1, 1]$ with cluster of points at both ends 1 and -1.

Notice that

$$z(0) = 1, \qquad z(\pi) = -1$$

So $z : [0, \pi] \longrightarrow [-1, 1]$ is decreasing

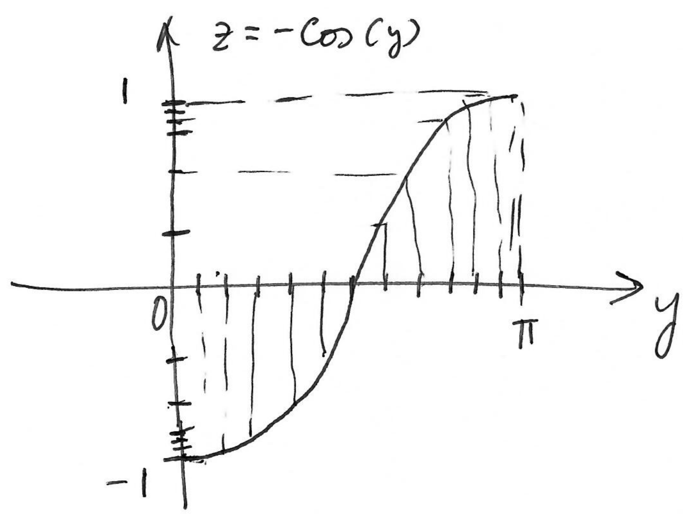
Since we want an increasing transformation,

we will use

$$z(y) = -\cos(y)$$

So the new transformation $z$ is

$$z : [0, \pi] \rightarrow [-1, 1]$$

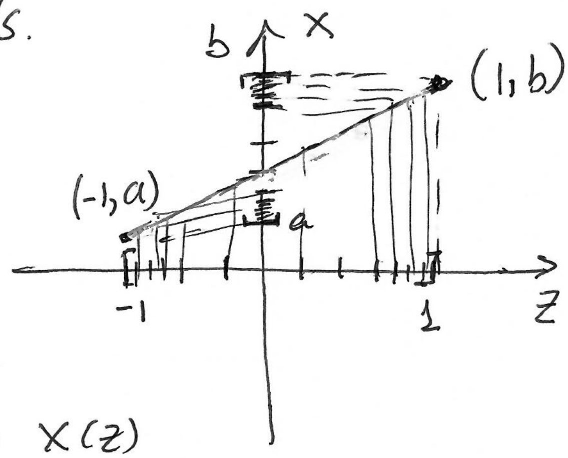


For an arbitrary interval $[a, b]$, we need a cluster of points near "a" and "b".

A linear transformation

$$X : [-1, 1] \longrightarrow [a, b]$$
$$z \longrightarrow X(z)$$

will be sufficient to preserve the clusters at both ends.




Defining $X(z)$

$$X(z) = mz + d, \qquad m = \frac{b-a}{2}$$

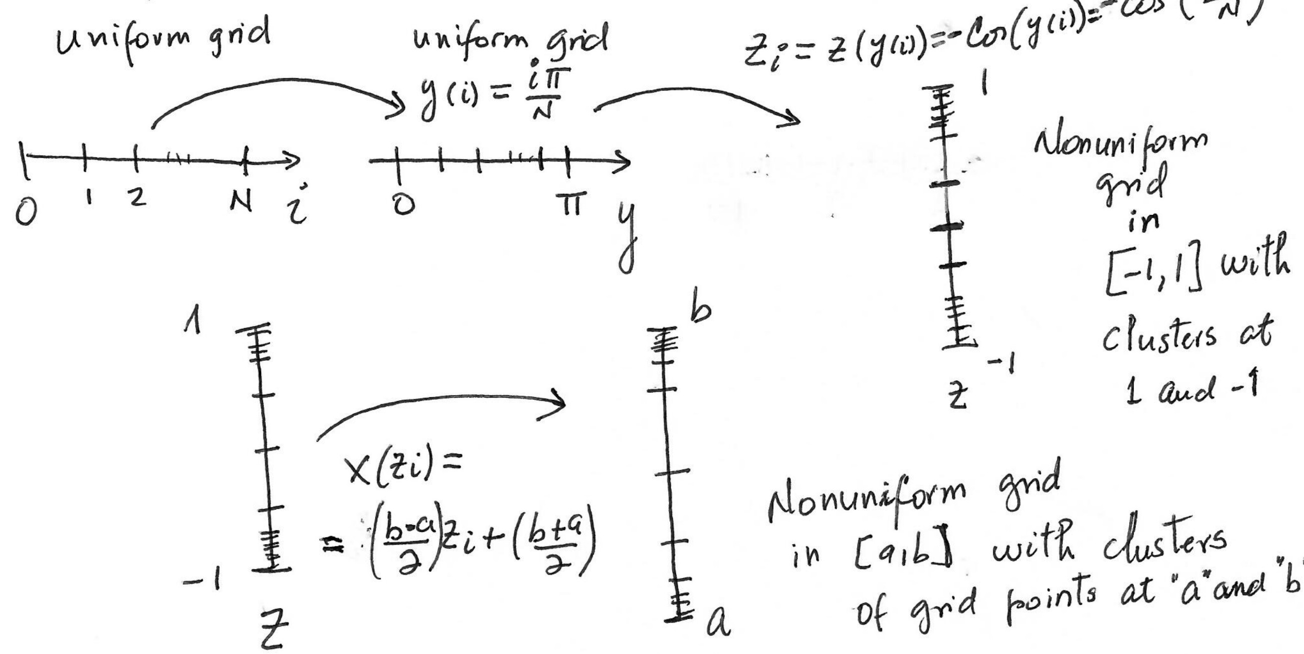
$$\Rightarrow \quad X(z) = \frac{b-a}{2} z + d$$

Using the point $(1, b)$

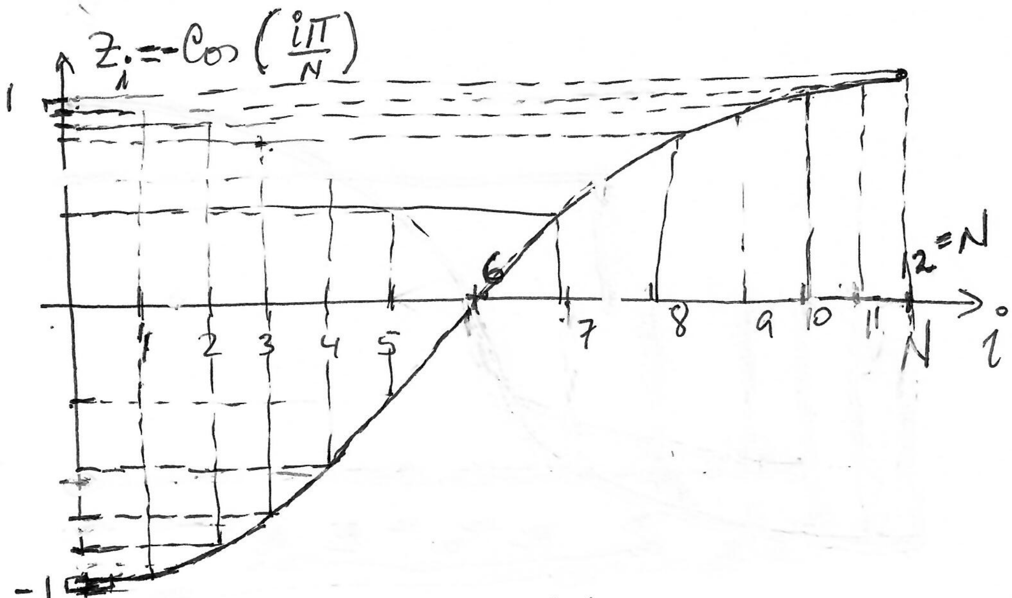
$$b = X(1) = \frac{b-a}{2} + d \Rightarrow d = \frac{b+a}{2}$$

$$\Rightarrow \quad \boxed{X(z) = \left(\frac{b-a}{2}\right) z + \frac{b+a}{2}}$$

So a Chebyshev grid in the interval $[a,b]$ can be obtained by composing the following transformations:




the partial result for $z_i$ is graphed as follows: (for $N=12$)




Final grid in interval $[a,b]$ with clusters of grid points at "a" and "b".

$$X_i = -\left(\frac{b-a}{2}\right)\left(\cos\left(\frac{i\pi}{N}\right)\right) + \frac{b+a}{2}, \quad i = 0,1,2,..,N.$$