# Convergence Theorems

Consider a finite difference method given by

$$\boxed{A\vec{v} = \vec{f},} \qquad A \; m\times m \text{ matrix.} \quad (1)$$

Spliting the matrix $A$ as

$$A = B - E, \qquad B, E \;\; m\times m \text{ matrices.} \quad (2)$$

System (1) can be written as

$$Bv = Ev + \vec{f}$$

If $B$ is invertible then,

$$v = B^{-1}Ev + B^{-1}\vec{f}$$

or

$$\boxed{\vec{v} = M\vec{v} + \vec{c}} \qquad (3)$$

Where $M = B^{-1}E$ and $\vec{c} = B^{-1}\vec{f}$.

A natural <u>iterative method</u> can be defined from (3) as

$$\boxed{\vec{v}^{(k+1)} = Mv^{(k-1)} + \vec{c},} \qquad k = 1, 2, \ldots \quad (4)$$

So starting with an initial guess $\vec{v}_0$,
we generate a sequence $\{\vec{v}^{(k)}\}_{k=1}^{\infty}$ using (4).

If B is invertible then the linear systems

$$A\vec{v} = \vec{f} \quad \text{and} \quad \vec{v} = M\vec{v} + \vec{c}$$

are equivalent, i.e.,

$v^*$ is a solution of (3)

$$\boxed{Av^* = \vec{f}}$$

if and only if $v^*$ is a fixed point of (4)

$$\boxed{\vec{v}^* = Mv^* + \vec{c}}$$

We have already found a sufficient condition for the convergence of (4) when $k \to \infty$ to a fixed point $\vec{v}^*$ of (3).

If $\|M\| < 1$, then the sequence $\{v^{(k)}\}_1^{\infty}$ of iterative method converges to the fixed point $v^*$ of (4).

## Convergent Matrices

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In studying iterative matrix techniques, it is of particular importance to know when [powers] of a matrix become small (that is, when all the entries approach zero). Matrices of [this type] are called *convergent*.

**Definition 7.16** We call an $n \times n$ matrix $A$ **convergent** if

$$\lim_{k \to \infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n.$$

**Example 4** Show that

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is a convergent matrix.

*Solution* Computing powers of $A$, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}, \quad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix},$$

and, in general,

$$A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}.$$

So $A$ is a convergent matrix because

$$\lim_{k \to \infty} \left(\frac{1}{2}\right)^k = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{k}{2^{k+1}} = 0.$$

Notice that the convergent matrix $A$ in Example 4 has $\rho(A) = \frac{1}{2}$, because $\frac{1}{2}$ is the [only] eigenvalue of $A$. This illustrates an important connection that exists between the spectral radius of a matrix and the convergence of the matrix, as detailed in the following result.



**Theorem 7.17** The following statements are equivalent.

**(i)** $A$ is a convergent matrix.

**(ii)** $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

**(iii)** $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

**(iv)** $\rho(A) < 1$.

**(v)** $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$.

The proof of this theorem can be found in [IK], p. 14.

## RCISE SET 7.2

**1.** Compute the eigenvalues and associated eigenvectors of the following matrices.

**a.** $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

**d.** $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **e.** $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ **f.** $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

**2.** Compute the eigenvalues and associated eigenvectors of the following matrices

<u>Lemma.</u> If $\rho(M)<1$ then

i) $(I-M)^{-1}$ exists.

ii) $(I-M)^{-1} = I+M+M^2+\cdots = \sum_{j=0}^{\infty} M^j$

<u>Proof.</u>

i) First notice that $\rho(M)<1$ implies

$\lambda=1$ is not an eigenvalue of . (*)

Now, we will show that $\mu=0$ is not an eigenvalue of $I-M$.

In fact,

$\vec{x}\neq 0$ is an eigenvector of $I-M$

$\Leftrightarrow$ There is $\mu\in\mathbb{C}$ such that

$(I-M)\vec{x} = \mu\vec{x} \Leftrightarrow M\vec{x} = (1-\mu)\vec{x}$

But, from (*) $1-\mu\neq 1 \Leftrightarrow \mu\neq 0$

and as a consequence $(I-M)^{-1}$ exists.

ii) Call

$$S_m = I + M \cdots + M^m$$

then $\quad M S_m = M + M^2 + \cdots + M^{m+1}$

Thus,

$$(I - M) S_m = S_m - M S_m = I - M^{m+1}$$

and

$$(I-M) \lim_{m \to \infty} S_m = \lim_{m \to \infty} \left( I - M^{m+1} \right) = I - \lim_{m \to \infty} M^{m+1} \overset{\substack{\text{Prev.}\\ \text{Theor}\\ \text{B-F.}}}{=} I$$

Therefore,

$$(I-M) \lim_{m \to \infty} S_m = I$$

or $\quad (I - M)^{-1} = \sum_{j=0}^{\infty} M^j$

A more useful condition can be given in terms of the spectral radius $\rho(M)$.

<u>Theorem 1.</u>

If the spectral radius $\rho(M)<1$ then, the fixed point problem $\vec{u}=M\vec{u}+\vec{c}$ has a unique solution for any $\vec{c}$.

<u>Proof.</u>-

If $\rho(M)<1$, then $\lambda=1$ is not an eigenvalue of $M$ or there is not any $\vec{v}\neq\vec{0}$ such that

$$M\vec{v}=\vec{v}$$

This is equivalent to say, that there is not any $\vec{v}\neq 0$ such that

$$(I-M)\vec{v}=\vec{v}-M\vec{v}=\vec{0}$$

Therefore, $I-M$ is invertible and $(I-M)\vec{v}=\vec{c}$ has a unique solution $v^*$ for any $\vec{c}$

$$(I-M)\vec{v}^*=\vec{c} \iff v^*=Mv^*+\vec{c}$$

which is equivalent to say that $\vec{v}=M\vec{v}+\vec{c}$ has a unique fixed point $v^*$ for any $\vec{c}$.

## Theorem 2.

For any arbitrary vector $\vec{v}^{\circ} \in \mathbb{R}^n$ the sequence $\{v^{(k)}\}_{k=1}^{\infty}$ defined by

$$v^{(k)} = M v^{(k-1)} + \vec{c}, \quad k \geq 1$$

converges to the unique solution $v^*$ (fixed point) of

$$\vec{v} = M\vec{v} + \vec{c}$$

if and only if $\rho(M) < 1$.

## Proof.-

For the proof we need these results:

a) $\rho(M) < 1 \Leftrightarrow \lim\limits_{k \to \infty} M^k \vec{v} = \vec{0}$, for any $\vec{v}$.

b) $\rho(M) < 1 \overset{\text{Lemma}}{\Longrightarrow} (I-M)^{-1} = I + M + M^2 + \cdots = \sum\limits_{j=0}^{\infty} M^j$

c) $\rho^k(M) \leq \|M^k\|$

d) Theorem 1 : $\rho(M) < 1 \Rightarrow (I-M)$ is invertible
$\Rightarrow (I-M)^{-1}$ is invertible.

Then, for any $\vec{c}$, there exists a unique soln. $\vec{v}^*$ such that
$(I-M)\vec{v}^* = \vec{c} \Leftrightarrow (I-M)^{-1}\vec{c} = v^*$.

$$\boxed{\rho(M)<1 \iff \lim_{k\to\infty} M^k = [0].}$$

## A particular Case

If $M$ is diagonalizable

$$M = P^{-1} D P \qquad P = \begin{pmatrix} \hat{v}_1 & \dots & \hat{v}_n \\ \downarrow & & \downarrow \end{pmatrix} \text{ eigenvectors}$$

$$M^k = P^{-1} D^k P. \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \text{ eigenvalues.}$$

$$D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}$$

$$\rho(M)<1 \iff |\lambda_i|<1, \quad i=1..,n$$

$$\iff D^k \xrightarrow[k\to\infty]{} [0] \qquad \iff M^k \xrightarrow[k\to\infty]{} [0]$$

Proof. ($\Rightarrow$) Convergence $\Rightarrow \rho(M) < 1$

Let $\vec{v}^*$ be the unique Soln. of

$$\vec{v} = M\vec{v} + \vec{c}. \qquad \text{It means } v^* = M\vec{v}^* + \vec{c} \qquad (6.1)$$

and $\vec{v}^k \to \vec{v}^*$, where $\vec{v}^{(k)} = M\vec{v}^{(k-1)} + \vec{c}$ (6.2)

Then, subtracting (6.1) from (6.2).

$$\vec{e}^k = \vec{v}^{(k)} - \vec{v}^* = M(\vec{v}^{(k-1)} - v^*) = M\,\vec{e}^{(k-1)}$$

$$= M(M\,\vec{e}^{(k-2)}) = M^2\vec{e}^{k-2} = \ldots M^k\vec{e}^0$$

Therefore,

$$\|M^k\vec{e}^0\| = \|\vec{e}^k\| = \|\vec{v}^{(k)} - \vec{v}^*\| \xrightarrow[k\to\infty]{\text{Hip}} 0$$

Since $\vec{e}^0 = \vec{v}^{(0)} - \vec{v}^*$ and $\vec{v}^0$ is arbitrary

$$\|M^k\| = \max_{\|\vec{x}\|\neq 0} \frac{\|M^k\vec{x}\|}{\|\vec{x}\|} \xrightarrow[k\to\infty]{} 0 \quad \overset{\text{B-F Thm 7.17}}{\Longrightarrow} \lim_{k\to\infty}\|M^k\| = 0$$

Therefore, $\rho^{(k)}(M) = \rho(M^k) \le \|M^k\| \xrightarrow[k\to\infty]{} 0$

$\Rightarrow \quad \rho(M) < 1$

$(\Leftarrow)$ $\rho(M) < 1 \Rightarrow$ Convergence



$$\vec{v}^{(k)} = M\vec{v}^{(k-1)} + \vec{c} = M(M\vec{v}^{(k-2)} + \vec{c}) + \vec{c}$$

$$= M^2\vec{v}^{(k-2)} + M\vec{c} + \vec{c}$$

$$\vdots$$

$$= \underbrace{M^k\vec{v}^0}_{\xrightarrow[(a)]{k\to\infty}\ 0} + \underbrace{\left(\sum_{j=0}^{k-1} M^j\right)\vec{c}}_{\xrightarrow{(b)\ \text{lemma}}\ (I-M)^{-1}\vec{c}}$$

$$\therefore \quad \lim_{k\to\infty} \vec{v}^{(k)} = (I-M)^{-1}\vec{c} \overset{(d)}{=} \vec{v}^*$$

Assuming that

$$M = B^{-1}E \text{ and } A = B - E, \text{ and } \vec{c} = B^{-1}\vec{f}.$$

then $v^*$ is also the solution of

$$A\vec{v} = \vec{f}.$$

In fact, if $\quad v^* = M\vec{v}^* + \vec{c}$

Then, $\quad \vec{v}^* = (B^{-1}E)\vec{v}^* + \vec{c}$

$$\Rightarrow \quad B\vec{v}^* = E\vec{v}^* + B\vec{c} = E\vec{v}^* + B(B^{-1}f)$$

$$\Rightarrow \quad (B-E)\vec{v}^* = \vec{f} \Rightarrow A\vec{v}^* = \vec{f}.$$

# Ideas for proofs of Lemma and Theorem

Lemma

$\rho(M) < 1 \Rightarrow (I-M)^{-1}$ exists

i)

$\lambda = 1$ not an eigenvalue $M \Rightarrow \mu = 0$ not an eigenv. of $I-M$.

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ii) Similar to geometric sequence proof of converg.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

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Thm 2

$(\rightarrow)$ Converg $\Rightarrow \rho(M) < 1$ $\longrightarrow \|\vec{v}^k - \vec{v}\| \underset{k\to\infty}{\longrightarrow} 0$ Hip

$$\underbrace{\rho^k(M) \le \|M^k\| \to 0} \Rightarrow \rho(M) < 1$$

and $\|M^k\| \to 0$ Since $\|M^k \vec{e}_0\| \to 0$ for any $\vec{e}_0$.

$(\leftarrow)$ Prove directly

$$\vec{v}^{(k)} \longrightarrow \vec{v}^*$$

Just developing $\vec{v}^{(k)} = M\vec{v}^{(k-1)} + \vec{C} = \dots$

Alternative: $\rho(M) < 1 \underset{(1)}{\Rightarrow}$ there exists $v^*$ s.t.

$v^* = Mv^* + \vec{C}$ unique soln.

(2) $\|M^k\| \underset{k\to\infty}{\longrightarrow} 0$.

$$\Rightarrow \|\vec{v}^{(k)} - v^*\| = \|\vec{e}^k\| = \dots = \|M^k \vec{e}^{(0)}\| \le \underset{\substack{\downarrow\, \rho(M)<1 \\ 0}}{\|M^k\|} \|\vec{e}^{(0)}\| \underset{k\to\infty}{\longrightarrow} 0.$$

**Thm** $A$ is S.D.D. $\Rightarrow$ $A$ is nonsingular.

**Proof.**

If $A$ is singular then there is $\vec{x} \neq \vec{0}$

s.t. $A\vec{x} = \vec{0}$.

Say $0 < |x_k| = \max\limits_{1 \le j \le n} |x_j|$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

From Hypothesi

$$\sum_{j=1}^{n} a_{kj} x_j = 0 \Rightarrow a_{kk} x_k = - \sum_{j \neq k} a_{kj} x_j$$

$$\Rightarrow |a_{kk}| \, |x_k| \le \sum_{j \neq k} |a_{kj}| \, |x_j|$$

$$\Rightarrow |a_{kk}| \le \sum_{j \neq k} |a_{kj}| \frac{|x_j|}{|x_k|} \le \sum_{j \neq k} |a_{kj}|$$

not S.D.D.

Consider $$A\vec{v}=\vec{f} \qquad (0.1)$$

and a decomposition of $A$ (splitting)

$$A=B-E.$$

Theorem. If $B$ is invertible

$\boxed{A\vec{v}=\vec{f}}$ is equivalent to $\boxed{\vec{v}=M\vec{v}+\vec{c}}$

where $M=B^{-1}E$, $\vec{c}=B^{-1}\vec{f}$

Proof.

$$
\begin{aligned}
A\vec{v}=\vec{f} &\Leftrightarrow (B-E)\vec{v}=\vec{f}\\
&\Leftrightarrow B\vec{v}=E\vec{v}+\vec{f}\\
&\Leftrightarrow \vec{v}=B^{-1}(E\vec{v})+B^{-1}f\\
&\Leftrightarrow \vec{v}=M\vec{v}+\vec{c}
\end{aligned}
$$

## Corollary.

If $A$ is S.D.D, then

a) The diagonal matrix $D$ corresponding to $A$ is invertible

b). $A\vec{v}=\vec{f}$ is equivalent to $\vec{v}=M_j\vec{v}+\vec{c}$ ($M_j$ is Jacobi matrix)

c) $\vec{v}=M_j\vec{v}+\vec{c}$ has a unique soln. $M_j = D^{-1}(L_T+U_T)$

Proof.

If $A$ is S.D.D.

then

$$D=\begin{pmatrix} a_{11} & \cdots & \cdots & 0 \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

is nonsingular since $a_{kk} \neq 0$, for all $k$.

then, according to previous theorem

$A\vec{v}=\vec{f}$ is equivalent to $\vec{v}=M_j\vec{v}+\vec{c}$

Also, $A\vec{v}=\vec{f}$ has a unique solution $v^*$, according to another previous theorem. in page 1.

As a consequence,

$v^*$ is also the unique soln (fixed point) of $v=M_j\vec{v}+\vec{c}$, or

$$v^* = M_j\vec{v}^* + \vec{c}.$$

**Theorem.** $A$ is S.D.D $\Rightarrow$ $\vec{v}^{(k)} = M_j v^{(k-1)} + \vec{c}$ converges to the unique soln. $\vec{v}^*$ of

$\vec{v} = M_j \vec{v} + \vec{c}$. in the $\|\cdot\|_\infty$. norm.

**Proof.** It's enough to prove that $\|M_j\|_\infty < 1$

In fact,

$$M_j = \begin{pmatrix} 0 & -a_{12}/a_{11} & \cdots & & -a_{1n}/a_{11} \\ -a_{21}/a_{22} & 0 & \cdots & \cdots & -a_{2n}/a_{22} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & -a_{n-1,n}/a_{n-1,n-1} \\ -a_{n1}/a_{nn} & \cdots & \cdots & -\dfrac{a_{n,n-1}}{a_{nn}} & 0 \end{pmatrix} = D^{-1}(L_T + U_T)$$

Then,

$$\|M_j\|_\infty = \max_{1\le i\le n} \sum_{\substack{j=1\\ j\ne i}}^{n} \left| -\frac{a_{ij}}{a_{ii}} \right| = \max_{1\le i\le n} \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{|a_{ij}|}{|a_{ii}|}$$

$$= \max_{1\le i\le n} \frac{1}{|a_{ii}|} \sum_{\substack{j=1\\ j\ne i}}^{n} |a_{ij}| \overset{\text{Hip. S.D.D.}}{<} 1$$

Application to our example,

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

$$\underset{\text{S.D.D.}}{A} \qquad \vec{x} = \vec{b}$$

Jacobi's method leads to

$$\begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

$$M_J \qquad \vec{x} = \vec{c}$$

Then,

$$\|M_J\| = \max_{1 \le i \le 3} \left( \sum_{j=1}^{3} |a_{ij}| \right) = \frac{2}{3} < 1$$

Therefore, Jacobi's method converges.