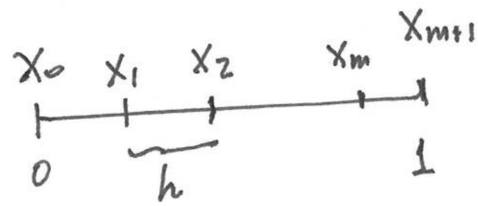
Consider the continuous Poisson problem 1-D.

$$u''(x) = f(x), \quad 0<x<1 \iff \boxed{u''(x) - f(x) = 0} \qquad (1)$$

and its 3-point centered difference finite difference scheme defined by

$$\boxed{\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} - f_i = 0} \qquad (2)$$

where



$$h = \frac{1}{m+1}$$

a) Show that the local trunation error of the approximation of (1) by (2) is given by

Cont. discrete

$$\boxed{\tau_i = \frac{h^2}{12} u''''(x_i) + O(h^4).} \qquad (3)$$

b) Assume that $D^2_{xxxx}(\cdot)$ is a $2^{\underline{nd}}$ order finite difference approximation of u_{xxxx}.
Also assume that $\bar{v}_i$ is a $2^{\underline{nd}}$ order approx. of $u(x_i)$. Then, Carefully show that the finite difference method defined as

$$\boxed{\left[\frac{\bar{v}_{i+1} - 2\bar{v}_i + \bar{v}_{i-1}}{h^2} - \frac{h^2}{12} D^2_{xxxx}(v_i)\right] - f_i = 0} \qquad (4)$$

approximates (1) with a local truncation error

$$\bar{\tau}_i = \frac{h^4}{360} u_{6x}(x_i) + O(h^6).$$

Proof.

a) Replacing u_i into (2)

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - f_i = u''(x_i) + \tau_i - f_i$$

$$= u''(x_i) - f_i + \tau_i = u''(x_i) - f_i + \frac{h^2}{12} u''''(x_i) + O(h^4)$$

Therefore,

$$\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - f_i\right] - \left[u''(x_i) - f_i\right] =$$

$$\frac{h^2}{12} u''''(x_i) + O(h^4).$$

and the proof is complete.

b) Assumig that

$$D^2_{xxxx}(v_i) - u_{xxxx}(x_i) = O(h^2).$$

b) Cont.

Subst. the exact Soln. $u(x_i)$ into (4).

$$\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{h^2}{12} D_{xxxx}(v_i)\right) - f_i$$

$$= u''(x_i) + \frac{h^2}{12} u_{xxxx}(x_i) + \frac{h^4}{360} u_{6x}(x_i)$$
$$- \frac{h^2}{12} D_{xxxx}(v_i) - f_i$$

$$= \left(u''(x_i) - f_i\right) - \frac{h^2}{12}\left(D_{4x}(v_i) - u_{4x}(x_i)\right)$$
$$+ \frac{h^4}{360} u_{6x}(x_i) + O(h^6).$$

$$= \left(u''(x_i) - f_i\right) - \frac{h^2}{12} O(h^2) + \frac{h^4}{360} u_{6x}(x_i) + O(h^6)$$

$$= \left(u''(x_i) - f_i\right) + O(h^4).$$

Therefore,

$$\left[\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{h^2}{12} D_{4x} v_i - f_i\right) - \left(u''(x_i) - f_i\right)\right] = O(h^4)$$

and the proof is complete!